Minimal Surfaces

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Introduction

This course aims to give an introduction to minimal surface theory, up to covering topics of current research. The content is inspired by many different references, main ones are the book by Colding–Minicozzi [13] and the notes by White [52] and Chodosh [8].

A classical motivating picture for studying minimal surfaces is a wire immersed in soapy water: after withdrawing it (and neglecting gravity), the soap film is expected to minimize its area among surfaces spanning the same boundary. Denote the wire by a smooth *Jordan curve* Γ (a simple closed curve) and a spanning surface by Σ, with $\partial\Sigma = \Gamma$.

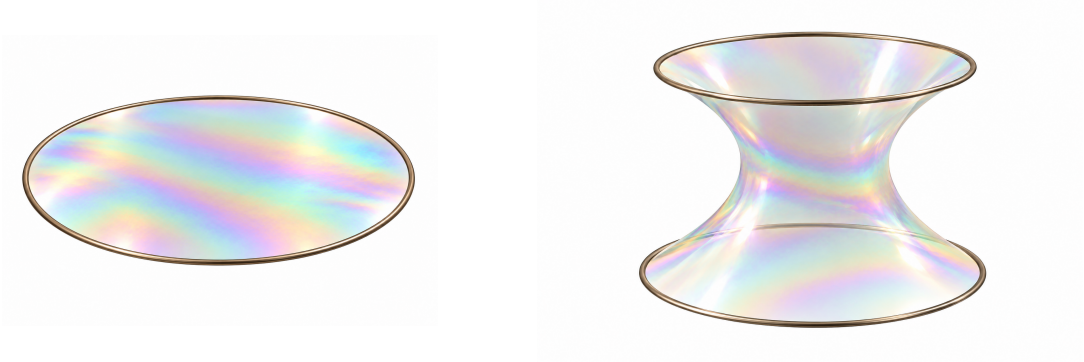


Figure 1: The Plateau problem: seek a least-area spanning surface Σ for a prescribed boundary curve Γ. A disk and a catenoid are familiar examples of minimal surfaces.

0.1 The Plateau problem and its history

Finding a surface with least area has a long history. In 1762, Lagrange [33] wrote down the equation for graphs of least area (the original memoir appeared in the volume dated 1760–1761). The problem of minimizing area with a fixed boundary curve Γ is named after Joseph Plateau, who investigated soap films (his findings were published in 1873 [44], after decades of research).

The classical Plateau problem was solved in the 1930s, independently by Jesse Douglas and Tibor Radó [17, 46]. Douglas received one of the first Fields Medals in 1936 for this work. An area-minimizing surface need not initially be given as an embedding: the construction may yield immersed surfaces and require an analysis of possible singularities and branch points.

For boundary curves in $\mathbb{R}^3$, work of Osserman and Gulliver in the 1970s ruled out interior branch points of area-minimizing disks [43, 25]. When $\Gamma \subset \partial K \subset \mathbb{R}^3$ for a compact

convex set K, Meeks–Yau [36] in 1982 established embeddedness results for least-area solutions.

Remark 0.1
It is an open problem whether branch points can occur at the (smooth) boundary.

Remark 0.2 - Branched minimal surfaces
The graph of the map $F\colon \mathbb{C} \to \mathbb{C}^2$ given by

$$F(z) = (z^2, z^3) \in \mathbb{C}^2$$

is a branched minimal surface.

Morrey [39] in 1948 generalized the problem to surfaces in Riemannian manifolds (M, g). There is also a higher-dimensional theory of area minimization. In the setting of geometric measure theory, the Federer–Fleming theory (1960) [20] provides least-mass representatives for appropriate integral homology classes, schematically

$$\sigma \in H_k(M; \mathbb{Z}) \quad \Longrightarrow \quad \text{an area-minimizing integral } k\text{-cycle representing } \sigma.$$

For codimension-one minimizers, write $\Sigma^n \subset M^{n+1}$. When $n \leq 6$ (ambient dimension at most 7), regularity theory yields smooth area-minimizing hypersurfaces in the interior, under the usual hypotheses; see also the foundational regularity work of Simons [50].

The subject subsequently developed in many directions: finding area-minimizing surfaces in new settings, constructing surfaces that are only critical points of the area functional (*minimal surfaces*), and understanding their geometry and topology.

0.2 Why study minimal surfaces?

Minimal surfaces are beautiful geometric objects in their own right. They also arise naturally in many problems as critical points or minimizers of geometric energies. Some examples are:

- **Scalar curvature and rigidity:** which Riemannian metrics on a torus T^n can have nonnegative scalar curvature? [48]
- **General relativity:** the positive mass theorem [49].
- **Knot theory:** a closed curve in $\mathbb{R}^3$ with total curvature less than 4π is unknotted [38]; counting minimal disks in hyperbolic 4-space $\mathbb{H}^4$ with prescribed knot as ideal boundary is related to knot invariants.
- **Image processing:** image denoising and optimal transport.
- **3-manifolds:** finding useful splittings, especially Heegaard splittings.
- **The Smale conjecture and diffeomorphism groups:** studying $\mathrm{Diff}(M)$ for a 3-manifold M.
- **Spectral theory.**

Some applications will be studied in the lectures; others are proposed as possible exam topics.

0.3 Course outline

1. Preliminaries in Riemannian geometry.
2. Foundations of minimal surface theory, including minimal surfaces in Euclidean space.
3. Minimal surfaces in positive curvature.
4. Minimal surfaces in negative curvature.
5. Existence of minimal surfaces.

Chapter 1

Preliminaries in Riemannian geometry

1.1 Smooth manifolds, tangent spaces, and vector fields

Let M^n be a smooth manifold and $p \in M$. Its tangent space T_pM may be defined as the vector space of velocities $\gamma'(0)$ of smooth curves γ passing through p. Equivalently, it consists of derivations at p, i.e., linear maps

$$v\colon C^\infty(M) \longrightarrow \mathbb{R}, \qquad v(fg) = f(p)v(g) + g(p)v(f).$$

Under this identification, the tangent vector $v = \gamma'(0)$ acts by

$$\gamma'(0)(f) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\gamma(t)).$$

The *tangent bundle* is the disjoint union

$$TM = \bigsqcup_{p\in M} T_pM.$$

A chart $(x^1, \dots, x^n)\colon U \subset M \to \mathbb{R}^n$ induces a basis $\left\{\partial_i|_p = \frac{\partial}{\partial x^i}\Big|_p\right\}_{i=1}^n$ of T_pM.

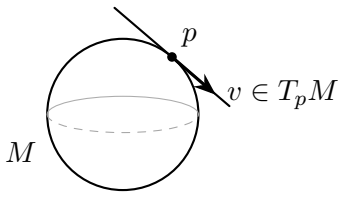


Figure 1.1: Schematic view of a tangent vector at a point of a manifold.

For a smooth map $F : M \to N$ between smooth manifolds, the *differential* at p is

$$\mathrm{d}F_p\colon T_pM \longrightarrow T_{F(p)}N, \qquad (\mathrm{d}F_p(v))(f) = v(f \circ F).$$

It satisfies the chain rule

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p.$$

The *cotangent space* is the dual $T_p^*M = (T_pM)^*$, and the *cotangent bundle* is T^*M. For $f \in C^\infty(M)$,

$$\mathrm{d}f_p \in T_p^*M, \qquad \mathrm{d}f_p(v) = v(f).$$

In local coordinates, the covectors $\mathrm{d}x^i|_p$ form the dual basis to the $\partial_i|_p$ and

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\mathrm{d}x^i.$$

Here and below, repeated upper and lower indices may be summed over, in Einstein notation.

A *smooth vector field* $X \in \mathfrak{X}(M) = \Gamma(TM)$ is a smooth section of the tangent bundle. Locally,

$$X = X^i\partial_i, \qquad X(f) = \mathrm{d}f(X).$$

Every smooth vector field has a local *flow* Φ_t^X, defined by

$$\frac{\partial}{\partial t}\Phi_t^X(p) = X(\Phi_t^X(p)), \qquad \Phi_0^X(p) = p.$$

If M is compact, or more generally if X has compact support, the flow is defined for all $t \in \mathbb{R}$, and each $\Phi_t^X \colon M \to M$ is a diffeomorphism.

1.2 Tensors and differential forms

An (r, s)-tensor field is a $C^\infty(M)$-multilinear map

$$T\colon \underbrace{\Gamma(T^*M) \times \cdots \times \Gamma(T^*M)}_{r \text{ factors}} \times \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{s \text{ factors}} \longrightarrow C^\infty(M).$$

We denote by $T^{(r,s)}(M)$ the space of (r, s)-tensor fields. In coordinates,

$$T = T^{i_1\cdots i_r}_{j_1\cdots j_s}\,\partial_{i_1} \otimes \cdots \otimes \partial_{i_r} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_s}.$$

In particular, vector fields are $(1, 0)$-tensor fields.

A *k-form* $\omega \in \Omega^k(M)$ is an alternating $(0, k)$-tensor. Thus

$$\omega(\ldots, X_i, \ldots, X_j, \ldots) = -\omega(\ldots, X_j, \ldots, X_i, \ldots).$$

The *wedge product* is the alternating tensor product: for $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^\ell(M)$,

$$(\alpha \wedge \beta)(X_1, \ldots, X_{k+\ell}) = \frac{1}{k!\ell!}\sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma)\alpha(X_{\sigma(1)}, \ldots, X_{\sigma(k)})\beta(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+\ell)}).$$

A manifold of dimension n is *orientable* if it admits a nowhere-vanishing smooth n-form.

1.3 Riemannian metrics, distance, and volume

A *Riemannian manifold* (M^n, g) is a smooth manifold endowed with a smooth, symmetric, positive-definite $(0, 2)$-tensor g, its Riemannian metric. It defines an inner product $\langle \cdot, \cdot \rangle_p$ on each T_pM. Locally,

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \qquad (g^{ij}) = (g_{ij})^{-1}.$$

The *Riemannian distance* is the infimum of lengths of piecewise smooth curves joining two points:

$$d(p,q) = \inf_{\substack{\gamma\colon [a,b]\to M \\ \gamma(a)=p,\,\gamma(b)=q}} \ell(\gamma), \qquad \ell(\gamma) = \int_a^b |\gamma'(t)|_g \mathrm{d}t.$$

If M is oriented, its *Riemannian volume form* $\mathrm{d}\,\mathrm{vol}_g$ takes value 1 on each positively oriented orthonormal basis $e_1, \dots, e_n$. In oriented coordinates,

$$\mathrm{d}\,\mathrm{vol}_g = \sqrt{\det(g_{ij})}\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n,$$

so integration becomes

$$\int_M f\,\mathrm{d}\,\mathrm{vol}_g = \int_{x(U)} (f \circ x^{-1}) \sqrt{\det(g_{ij})}\,\mathrm{d}x^1 \cdots \mathrm{d}x^n$$

when f is supported inside a positively oriented coordinate chart U.

Theorem 1.1 - Stokes' theorem
For a smooth oriented n-manifold M (possibly with boundary) and a compactly supported $(n-1)$-form ω,

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega,$$

using the induced orientation on ∂M.

1.4 The Levi–Civita connection

A *connection* on the tangent bundle assigns to vector fields X, Y a new vector field $\nabla_X Y$. The *Levi–Civita connection* is characterized by the following conditions:

(i) ∇ is $C^\infty(M)$-linear in X, $\mathbb{R}$-linear in Y, and satisfies the Leibniz rule

$$\nabla_X(fY) = X(f)Y + f\nabla_X Y.$$

(ii) *Torsion-free:*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

(iii) *Metric compatible:*

$$X\big(g(Y,Z)\big) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

In local coordinates the *Christoffel symbols* Γ_{ij}^k are defined by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k.$$

On Euclidean space $\mathbb{R}^n$, the Levi–Civita connection is the ordinary directional derivative:

$$\nabla^{\mathbb{R}^n}_X Y = D_X Y = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} Y(p + tX(p)).$$

Notation 1.1
Along a curve $\gamma\colon [a,b] \to M$, covariant differentiation of a vector field Y along γ is written

$$\frac{\nabla Y}{\mathrm{d}t} = \nabla_{\gamma'(t)} Y.$$

For a vector field Y *along* γ we abbreviate its first and second covariant derivatives by

$$Y' := \frac{\nabla Y}{\mathrm{d}t}, \qquad Y'' := \frac{\nabla}{\mathrm{d}t} Y'.$$

Thus a prime on a vector field along γ means *covariant* differentiation; in $\mathbb{R}^n$ this agrees with the usual derivative. The connection also extends, via the Leibniz rule, to tensor fields.

1.5 Curvature

The *Riemann curvature tensor* R of (M^n, g) is defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

We also use the associated $(0,4)$-tensor, with the following convention for the last two arguments:

$$R(X,Y,Z,W) := g(R(X,Y)W, Z).$$

The *sectional curvature* of the plane spanned by linearly independent vectors $X, Y \in T_pM$ is

$$\mathrm{Sec}(X,Y) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2 - g(X,Y)^2}.$$

The *Ricci curvature* is the trace

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i),$$

where $(e_i)_{i=1}^n$ is an orthonormal basis of T_pM. The *scalar curvature* is

$$\mathrm{Scal} = \mathrm{tr}_g \,\mathrm{Ric} = \sum_{i \neq j} \mathrm{Sec}(e_i, e_j).$$

The curvature tensor satisfies the following symmetries and identities:

$$\begin{aligned} R(X,Y,Z,W) &= R(Z,W,X,Y) = -R(X,Y,W,Z), \\ R(X,Y)Z + R(Y,Z)X + R(Z,X)Y &= 0 \qquad \text{(first Bianchi identity)}, \\ (\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) &= 0 \qquad \text{(second Bianchi identity)}. \end{aligned}$$

In the last identity, R is viewed as a $(1,3)$-tensor and the equality holds after applying both sides to a fourth vector.

Example 1.2 - Space forms
The model spaces $\mathbb{R}^n$, S^n (round unit sphere), and $\mathbb{H}^n$ have constant sectional curvature $\kappa = 0, 1, -1$, respectively. More generally, on a manifold of constant sectional curvature κ,

$$\mathrm{Ric} = (n-1)\kappa\, g, \qquad \mathrm{Scal} = n(n-1)\kappa.$$

These models are, up to isometry, the complete simply connected space forms in their respective curvature classes.

1.6 Operators on manifolds

The metric induces the *musical isomorphisms*

$$\begin{aligned} \flat\colon TM &\longrightarrow T^*M, \qquad X^\flat(Y) = g(X,Y), \\ \sharp\colon T^*M &\longrightarrow TM, \qquad g(\omega^\sharp, Y) = \omega(Y). \end{aligned}$$

In coordinates,

$$(X^\flat)_j = g_{ij}X^i, \qquad (\omega^\sharp)^i = g^{ij}\omega_j.$$

For $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$, we define

$$\begin{aligned} \nabla f &:= (\mathrm{d}f)^\sharp && \text{(gradient)}, \\ \operatorname{div} X &:= \operatorname{tr}(\nabla X) = \sum_{i=1}^n g(\nabla_{e_i} X, e_i) && \text{(divergence)}, \\ \nabla^2 f &:= \nabla(\mathrm{d}f), \quad \nabla^2 f(X,Y) = g(\nabla_X \nabla f, Y) && \text{(Hessian)}, \\ \Delta f &:= \operatorname{div}(\nabla f) = \operatorname{tr}_g(\nabla^2 f) && \text{(Laplacian)}. \end{aligned}$$

Here (e_i) is a local orthonormal frame. In particular, the convention for Δ is $\Delta = \operatorname{div}\nabla$.

Theorem 1.2 - Divergence theorem
Let (M^n, g) be oriented, with smooth boundary, and let X be a compactly supported smooth vector field (smooth up to the boundary). If ν is the outward-pointing unit normal to ∂M, then

$$\int_M \operatorname{div} X \,\mathrm{d}\operatorname{vol}_g = \int_{\partial M} g(X,\nu)\,\mathrm{d}\operatorname{vol}_{\partial M}.$$

Proof. By inserire Cartan's formula for the Lie derivative and the fact that a top-degree form is closed,

$$\mathrm{d}(\iota_X \mathrm{d}\operatorname{vol}_g) = \mathcal{L}_X \mathrm{d}\operatorname{vol}_g = (\operatorname{div} X)\,\mathrm{d}\operatorname{vol}_g.$$

Moreover, on ∂M one has

$$(\iota_X \mathrm{d}\operatorname{vol}_g)|_{\partial M} = g(X,\nu)\,\mathrm{d}\operatorname{vol}_{\partial M}.$$

The assertion follows from Stokes' theorem. □

Remark 1.3
An alternative proof in local coordinates, using the divergence formula and Gauss's theorem in $\mathbb{R}^n$, is left as an exercise.

1.7 Submanifolds

We use $\Sigma^k \subset M^n$ for a k-dimensional submanifold of an n-dimensional ambient manifold. For hypersurfaces we instead write $\Sigma^n \subset M^{n+1}$.

For a k-dimensional submanifold S^k, we use $\operatorname{vol}(S)$ and $|S|$ interchangeably for its k-dimensional Riemannian volume (measure). Thus $|\Sigma|$ denotes k-dimensional volume when $\Sigma^k \subset M^n$, whereas $|\partial\Sigma|$ denotes the $(k-1)$-dimensional measure of its boundary.

For a smooth map $F\colon \Sigma^k \to M^n$:

- F is an *immersion* if $\mathrm{d}F_p$ is injective for every $p \in \Sigma$;

- F is an *embedding* if it is an immersion and a homeomorphism onto its image (with the subspace topology);
- F is an *isometry* if $F^* g_M = g_\Sigma$, where g_Σ, g_M are metrics on Σ, M.

From now on, unless otherwise stated, we assume that $F\colon \Sigma^k \to M^n$ is an isometric immersion. As usual, we identify Σ with its image locally and regard tangent vector fields as fields along the immersion. The intrinsic Levi–Civita connection is the tangential projection of the ambient one:

$$\nabla^\Sigma_X Y = (\nabla^M_X Y)^\top.$$

Notation 1.4
In this subsection, for brevity, write $\nabla := \nabla^\Sigma$ and $D := \nabla^M$.

Definition 1.3 - Second fundamental form
The *second fundamental form* is the normal-bundle-valued symmetric bilinear form

$$A(X,Y) := (D_X Y)^\perp \in T\Sigma^\perp.$$

For tangent fields X, Y, it satisfies

$$A(X,Y) = A(Y,X), \qquad D_X Y = \nabla_X Y + A(X,Y).$$

The (unnormalized) *mean curvature vector* is

$$\vec{H} := \operatorname{tr}_{T\Sigma} A = \sum_{i=1}^{k} A(e_i, e_i).$$

Proposition 1.4 - Gauss equation
For tangent vectors $X, Y, Z, W \in T_p\Sigma$,

$$R^\Sigma(X,Y,Z,W) = R^M(X,Y,Z,W) + g(A(X,Z), A(Y,W)) - g(A(X,W), A(Y,Z)).$$

Proposition 1.5 - Codazzi equation
For tangent fields X, Y, Z,

$$(\nabla_X A)(Y,Z) - (\nabla_Y A)(X,Z) = \big(R^M(X,Y)Z\big)^\perp,$$

where ∇A uses the induced tangent and normal connections.

Hypersurfaces

For a hypersurface $\Sigma^n \subset M^{n+1}$, we say Σ is *two-sided* if there exists a global unit normal field $N\colon \Sigma \to T\Sigma^\perp$. Once N is chosen, define

$$\mathrm{II}(X,Y) := -g(A(X,Y), N), \qquad H := -g(\vec{H}, N).$$

Thus $H = \operatorname{tr}_{T\Sigma} \mathrm{II}$.

Example 1.5 - Round sphere
For $S^n \subset \mathbb{R}^{n+1}$ with the outward unit normal $N(x) = x$,

$$A(X,Y) = -g(X,Y)N, \qquad \mathrm{II}(X,Y) = g(X,Y),$$

and hence $\vec{H} = -nN$ and $H = n$.

Definition 1.6 - Shape operator
For a chosen unit normal N, the *shape operator* is

$$S_N \colon T_p\Sigma \longrightarrow T_p\Sigma, \qquad S_N(X) := -(D_X N)^\top.$$

It satisfies

$$g(S_N(X), Y) = g(A(X,Y), N) = -\,\mathrm{II}(X,Y).$$

The *principal curvatures* (with this convention) are the eigenvalues of S_N.

Remark 1.6
With the signs above, the eigenvalues of S_N are the negatives of those of the quadratic form II with respect to g. In particular, $S_N = -\,\mathrm{id}$ for the outward-oriented unit sphere.

1.8 Geodesics

Let (M^n, g) be a complete Riemannian manifold (for example, $S^3 \subset \mathbb{R}^4$ with the round metric). Given $p, q \in M$, what is the shortest path between them?

For a piecewise smooth curve $\gamma \colon I = [a,b] \to M$, define its *length* by

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\mathrm{d}t, \qquad |\gamma'(t)| = \sqrt{g(\gamma'(t), \gamma'(t))}.$$

Given two points p, q, we seek a curve γ between them minimizing $\ell(\gamma)$.

Rather then minimizing directly the length functional, it is often convenient to minimize the *energy*

$$E(\gamma) = \frac{1}{2}\int_a^b |\gamma'(t)|^2 \mathrm{d}t.$$

Indeed, by Cauchy-Schwarz, one has:

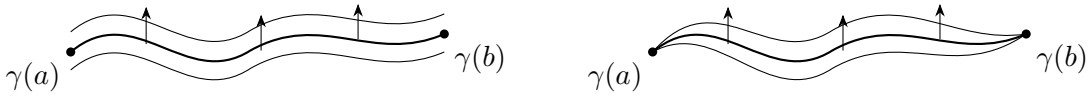
$$\ell(\gamma)^2 \leq 2(b-a)E(\gamma),$$

with equality if and only if γ has constant speed. Since a repametrization of a curve γ does *not* change its image, we can always find a diffeomorphism $\sigma \colon I \to I$ so that $\gamma(\sigma(t))$ has constant speed. Thus energy minimizers correspond to length minimizers parametrized with constant speed. One of the main advantages of working with E instead of ℓ is its better analytical structure, which makes it particularly suitable for proving the existence of minimizers.

Definition 1.7
A smooth map $F \colon I \times (-\varepsilon, \varepsilon) \to M$ is a *variation* of γ if $F(t,0) = \gamma(t)$. It is *proper* (fixed endpoints) if $F(a,s) = \gamma(a)$ and $F(b,s) = \gamma(b)$ for every s. Its associated

variational field is

$$X(t) = \left.\frac{\partial F(t,s)}{\partial s}\right|_{s=0}.$$



Variation with moving endpoints — Proper variation (fixed endpoints)

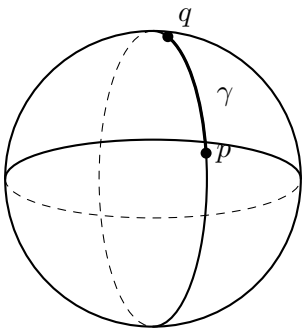
Figure 1.2: Two variations of a curve. On the left the endpoints move; on the right they remain fixed (a proper variation).

Proposition 1.8 - First variation of energy
For a variation F of γ with variational field X,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}E(F(\cdot,s))\right|_{s=0} = -\int_a^b \left\langle X, \frac{\nabla}{\mathrm{d}t}\gamma' \right\rangle \mathrm{d}t + \langle X(b), \gamma'(b)\rangle - \langle X(a), \gamma'(a)\rangle.$$

Definition 1.9
A smooth curve $\gamma\colon I \to M$ is a *geodesic* if $\frac{\nabla}{\mathrm{d}t}\gamma' = 0$ on I. Equivalently, it is a critical point of the energy with respect to proper variations. A *(minimizing) geodesic segment* is a geodesic which minimizes the length between its endpoints.

Example 1.7 - Geodesics on S^2
Great circles are geodesics on the round sphere. A long arc of a great circle need not be length minimizing between its endpoints.



γ is a geodesic segment: an arc of a great circle on the round sphere.

Figure 1.3: The round sphere S^2 with a geodesic γ joining two points p and q. Great circles are exactly the geodesics of the round metric.

Remark 1.8
Every minimizing segment is a geodesic, but the converse need not hold globally. Every geodesic is locally a minimizing segment.

Proposition 1.10 - Local existence and uniqueness
For every $p \in M$, there is a neighborhood U_p of $0 \in T_pM$ such that for any $v \in U_p$, there is a unique geodesic starting at p with initial velocity v, defined at least until time 1. The exponential map is $\exp_p(v) = \gamma_v(1)$.

Remark 1.9
The previous result is essentially the local existence for solutions to a sistem of ODEs. What can be said about the *global* existence?

Definition 1.11
Let M be a manifold. A connection ∇ on M is *geodesically complete* if the exponential map $\exp_p$ is defined on the full tangent space at $p \in M$. Equivalently, every maximal geodesic $\gamma(t)$ in M needs to be defined for all times $t \in \mathbb{R}$. A Riemannian manifold M is *geodesically complete* if its Levi-Civita connection is.

Theorem 1.12 - Hopf-Rinow
Let M be a connected Riemannian manifold. The following are equivalent:

1. *M is geodesically complete,*
2. *Given a subset $K \subset M$ then*

$$K \text{ compact} \iff K \text{ closed and bounded,}$$

3. *M is complete.*

Proposition 1.13 - Second variation of energy
Suppose γ is a geodesic and F is a variation of γ with variational field X. With the curvature convention used in these notes,

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(F(\cdot,s))\bigg|_{s=0} &= \int_a^b (|X'|^2 - \langle R(X,\gamma')\gamma', X\rangle)\mathrm{d}t + \left[\langle \nabla_s \partial_s F|_{s=0}, \gamma'\rangle\right]_a^b \\ &= -\int_a^b \langle X, X'' + R(X,\gamma')\gamma'\rangle \,\mathrm{d}t + \langle X, X'\rangle|_a^b + \left[\langle \nabla_s \partial_s F|_{s=0}, \gamma'\rangle\right]_a^b .\end{aligned}$$

For a proper variation

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}E(F(\cdot,s))\bigg|_{s=0} = -\int_a^b \langle X, X'' + R(X,\gamma')\gamma'\rangle \,\mathrm{d}t. \tag{1.1}$$

both boundary terms vanish.

Definition 1.14
The *Jacobi operator* along a geodesic is

$$J(X) = X'' + R(X,\gamma')\gamma'.$$

A field in the kernel of J is called a *Jacobi field*.

Theorem 1.15 - Myers [40]
Let (M^n, g) be complete and suppose

$$\mathrm{Ric} \geq (n-1)\kappa\, g \qquad \text{for some } \kappa > 0.$$

Then

1. *M is compact and*
$$\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{\kappa}};$$
2. *$\pi_1(M)$ is finite.*

Idea. If we are able to construct a proper variation F then thanks to 1.1 and the bound on the curvature we could infer the desired result. Having only a bound on Ric and not on $R(\cdot,\cdot)$, we need to "average". ⌟

Proof. For (i), let $\gamma\colon [0,\ell] \to M$ be a unit-speed geodesic. It suffices to show that if $\ell > \pi/\sqrt{\kappa}$, then $\gamma|_{[0,\ell]}$ is not minimizing.

Choose an orthonormal basis $E_\alpha(0)$, $\alpha = 1,\dots,n-1$, of $\gamma'(0)^\perp \subset T_pM$, and parallel-transport[1] it along γ; thus $E'_\alpha = 0$. Set $c = (\pi/\ell)^2 < \kappa$ and define

$$\mathrm{sn}_c(t) = \frac{1}{\sqrt{c}}\sin(\sqrt{c}\,t), \qquad V_\alpha(t) = \mathrm{sn}_c(t)E_\alpha(t), \qquad \mathrm{sn}_c'' + c\,\mathrm{sn}_c = 0.$$

The proper variations $F_\alpha(t,s) = \exp_{\gamma(t)}(sV_\alpha(t))$ have second variations whose sum equals

$$\begin{aligned}
\sum_{\alpha=1}^{n-1} \frac{\mathrm{d}^2}{\mathrm{d}s^2} E(F_\alpha(\cdot,s))\Big|_{s=0} &= \sum_{\alpha=1}^{n-1} \int_0^\ell \Big(|V_\alpha'|^2 - \langle R(V_\alpha,\gamma')\gamma', V_\alpha\rangle \Big)\,\mathrm{d}t \\
&= \int_0^\ell \Big((n-1)(\mathrm{sn}_c')^2 - \mathrm{sn}_c^2\,\mathrm{Ric}(\gamma',\gamma') \Big)\,\mathrm{d}t \\
&\leq (n-1)\int_0^\ell \big((\mathrm{sn}_c')^2 - \kappa\,\mathrm{sn}_c^2 \big)\mathrm{d}t \\
&= (n-1)\int_0^\ell (c-\kappa)\,\mathrm{sn}_c^2\,\mathrm{d}t < 0.
\end{aligned}$$

Here integration by parts and $\mathrm{sn}_c(0) = \mathrm{sn}_c(\ell) = 0$ give $\int(\mathrm{sn}_c')^2 = c\int \mathrm{sn}_c^2$. Thus at least one proper variation has negative second variation, contradicting minimizing behavior. Completeness and Hopf–Rinow now imply compactness and the diameter bound.

For (ii), let $\pi : \widetilde{M} \to M$ be the universal cover equipped with the pullback metric $\widetilde{g} = \pi^* g$. The cover is complete and satisfies $\widetilde{\mathrm{Ric}} \geq (n-1)\kappa\widetilde{g}$. By (i), $\widetilde{M}$ is compact. A compact universal cover has only finitely many sheets over M, so $\pi_1(M)$ is finite. □

Remark 1.10
With this result we have a nice example of a possible use of the theory of minimal surfaces on a manifold. The theorem uses geodesics, i.e. one dimensional minimal surfaces, to infer global geometric and topological information about a manifold.

[1] Namely, $E'_\alpha(t) = \frac{\nabla}{dt}E_\alpha = 0$. Writing it down in coordinates will actually give a ODE.

1.9 Exercises

Exercise 1.11 - Mean curvature of graphs
Let $f\colon \Omega \to \mathbb{R}$ be a smooth function with $\Omega \subset \mathbb{R}^n$ smooth open set. Let $\Sigma = \operatorname{graph} f \subset \mathbb{R}^{n+1}$ be the graph of f with metric tensor g inherited by $\mathbb{R}^{n+1}$.

(i) Show that Σ is a hypersurface in $\mathbb{R}^{n+1}$, by finding a global chart (Ω, φ) for Σ.

(ii) Denoting by ∂_i and ∇ the usual differentiation on $\mathbb{R}^n$, by N the upward unit normal field of Σ, and by II_N and H the second fundamental form and the mean curvature of Σ in direction N respectively. Prove that

$$g_{ij} = \delta_{ij} + \partial_i f \partial_j f, \qquad N = \frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}, \qquad \mathrm{II}_N = \frac{\nabla^2 f}{\sqrt{1+|\nabla f|^2}},$$

$$H = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{(\nabla^2 f)(\nabla f, \nabla f)}{\left(\sqrt{1+|\nabla f|^2}\right)^3}.$$

Exercise 1.12 - Two-sidedness of hypersurfaces
Show that every properly embedded hypersurface Σ^n in $\mathbb{R}^{n+1}$ is two-sided.

Exercise 1.13 - Divergence and Laplacian
Let (M^n, g) be a Riemannian manifold.

(i) Prove the expression of the divergence with respect to the metric in coordinates

$$\operatorname{div}(X) = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} X^k\right).$$

(ii) Prove the expression of the Laplacian with respect to the metric in coordinates

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l f\right).$$

(iii) Show that, if $f \in C_c^\infty(M)$ and $\Delta f \geq 0$, then f is constant.

Exercise 1.14 - Normal frame, normal chart, exponential normal chart
Let us fix a Riemannian manifold (M^n, g) and a point $p \in M$. Let us set $\{(e_i)|_p\}_{i=1}^n$ an orthonormal basis of T_pM.

(i) Show that there exists a local orthonormal frame $\{e_i\}_{i=1}^n$ of TM around p such that $e_i(p) = (e_i)|_p$, and $(\nabla_{e_i} e_j)(p) = 0$ for every $1 \leq i, j \leq n$. Notice that then, in such a frame, the Christoffel symbols are zero at p, i.e., $\Gamma_{ij}^k(p) = 0$ for every $1 \leq i, j, k \leq n$. We will call such a frame a *normal frame* around p.

(ii) Show that there exists a chart (U, φ) around p such that $\partial_i(p) = (e_i)|_p$ and $(\nabla \partial_i)(p) = 0$ for every $1 \leq i \leq n$. Show that if in this chart we have $g = g_{ij} dx^i \otimes dx^j$, then $g_{ij}(p) = \delta_{ij}(p)$, $\partial_k g_{ij}(p) = 0$, and $\Gamma_{ij}^k(p) = 0$ for every $1 \leq i, j, k \leq n$. We will call such a chart a *normal chart* around p. Notice that one example of such a chart is the *exponential normal chart*, that gives rise to *exponential normal coordinates*.

Exercise 1.15 - Warped products $I \times_f N$
Let (N^n, g) be a Riemannian manifold and let $I \subset \mathbb{R}$ be an interval. Given $f \in C^\infty(I)$ with $f \neq 0$ on I, we define the *warped product* manifold $M = I \times_f N$ to be the $(n+1)$-dimensional manifold $I \times N$ endowed with the warped metric

$$\overline{g} = dt^2 + f^2(t)g.$$

Let $e_1, \dots, e_n$ be a local orthonormal frame on an open set $U \subset N$. Assume also that $\{e_i\}$ is normal at some $x \in U$. Let also

$$\overline{e}_0 = \partial_t, \qquad \overline{e}_i = \frac{1}{f} e_i.$$

(i) Observe that $\overline{e}_0, \overline{e}_1, \dots, \overline{e}_n$ is an orthonormal frame on $I \times U$.

(ii) If R and $\overline{R}$ are the Riemann tensor on N and $I \times_f N$ respectively, prove that

$$\overline{R}_{0ijk} = \overline{R}_{i0jk} = 0, \qquad \overline{R}_{i0j0} = -\frac{f''}{f}\delta_{ij}, \qquad \overline{R}_{ijkl} = \frac{1}{f^2}R_{ijkl} - \left(\frac{f'}{f}\right)^2 (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

for any $i, j, k, l = 1, \dots, n$.

(iii) If N has constant sectional curvature $c \in \mathbb{R}$, deduce that

$$\overline{\sec}(\nabla t \wedge \overline{e}_i) = -\frac{f''}{f}, \qquad \overline{\sec}(\overline{e}_i \wedge \overline{e}_j) = \frac{c - (f')^2}{f^2}$$

for any $i, j = 1, \dots, n$ with $i \neq j$.

Exercise 1.16 - Radial models of constant sectional curvature
A *radial model of constant sectional curvature* $c \in \mathbb{R}$ is

$$\left((0, T) \times \mathbb{S}^{n-1}, \langle \cdot, \cdot \rangle := \mathrm{d}r^2 + \mathrm{sn}_c(r)^2 \mathrm{d}\theta^2\right),$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere of radius one and $\mathrm{d}\theta^2$ is the standard metric on it; the function $\mathrm{sn}_c(r)$ is the positive solution of the Jacobi equation

$$\begin{cases} \mathrm{sn}_c''(r) + c\,\mathrm{sn}_c(r) = 0, \\ \mathrm{sn}_c(0) = 0, \\ \mathrm{sn}_c'(0) = 1; \end{cases}$$

and $T := \sup\{\varrho \ : \ \mathrm{sn}_c(r) > 0 \ \forall r \in (0, \varrho)\}$.

(i) Prove that

$$\mathrm{sn}_c(r) = \begin{cases} \frac{1}{\sqrt{c}} \sin(\sqrt{c}\, r) & \text{if} \quad c > 0, \quad \text{then} \quad T = \frac{\pi}{\sqrt{c}}, \\ r & \text{if} \quad c = 0, \quad \text{then} \quad T = +\infty, \\ \frac{1}{\sqrt{-c}} \sinh(\sqrt{-c}\, r) & \text{if} \quad c < 0, \quad \text{then} \quad T = +\infty. \end{cases}$$

(ii) Show that, given $r(\cdot) := d(0, \cdot)$ the distance from the origin in this model, we have

$$\nabla^2 r(\cdot, \cdot) = \frac{\mathrm{sn}_c'(r)}{\mathrm{sn}_c(r)} \left(\langle \cdot, \cdot \rangle - \mathrm{d}r \otimes \mathrm{d}r(\,\cdot, \cdot)\right).$$

(iii) Show that these models have actually constant sectional curvature c.

Exercise 1.17 - The round sphere
For $n \geq 2$, let $\mathbb{S}^n_r \coloneqq \{x \in \mathbb{R}^{n+1} \ : \ |x| = r\}$ be the round sphere of radius r, with the metric g inherited by $\mathbb{R}^{n+1}$ and the outward unit normal $N(x) = x/r$.

(i) Show that $\mathbb{S}^n_r$ has second fundamental form $A(X,Y) = -\frac{1}{r}\langle X, Y\rangle N$, second fundamental form in the normal direction $\mathrm{II}(X,Y) = \frac{1}{r}\langle X, Y\rangle$, and shape operator $S_N = -\frac{1}{r}\mathrm{id}$. Deduce that the vector and scalar mean curvatures of $\mathbb{S}^n_r$ are equal to $\vec{H} = -\frac{n}{r}N$ and $H = \frac{n}{r}$.

(ii) Show that the curvature tensor of $\mathbb{S}^n_r$ is

$$R(X,Y,Z,W) = \frac{1}{r^2}\big(\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle\big).$$

Deduce that $\mathbb{S}^n_r$ has constant sectional curvature $\frac{1}{r^2}$, Ricci curvature $\mathrm{Ric} = \frac{n-1}{r^2}\, g$, and scalar curvature $\mathrm{Scal} = \frac{n(n-1)}{r^2}$.

Chapter 2

Classical theory for minimal surfaces

2.1 The minimal surface equation

Let (M^n, g) be a complete Riemannian manifold and $\Sigma^k \subset M^n$ an embedded[1] submanifold.

Definition 2.1
For a smooth vector field X along Σ, define its divergence along Σ by

$$\operatorname{div}_\Sigma X = \sum_{i=1}^{k} g(\nabla_{e_i} X, e_i) = \operatorname{tr}_{T\Sigma}(\nabla X),$$

where $\{e_1, \dots, e_k\}$ is an orthonormal basis of $T_p\Sigma$.

Remark 2.1 - Tangential vs. ambient divergence
Consider $\Sigma^k \subset M^n$ as above. Suppose that X is a smooth vector field defined on an open neighborhood of Σ in M. If $\{e_1, \dots, e_k\}$ is an orthonormal basis of $T_p\Sigma$, completed to an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM, then

$$\operatorname{div}_M X = \operatorname{div}_\Sigma X + \sum_{\alpha=k+1}^{n} g(\nabla_{e_\alpha} X, e_\alpha).$$

In particular, the tangential divergence only involves derivatives along directions tangent to Σ. Moreover, if X is defined only along Σ, we can compute $\operatorname{div}_\Sigma X$ using any smooth extension of X to a neighborhood in M. The result is independent of the chosen extension.

Achtung: In general, $\operatorname{div}_\Sigma X$ does not coincide with the ambient divergence $\operatorname{div}_M X$, since only tangential derivatives are taken into account.

In particular, if Σ is a hypersurface with unit normal ν, then

$$\boxed{\operatorname{div}_\Sigma X = \operatorname{div}_M X - g(\nabla_\nu X, \nu).}$$

[1] Much of the discussion also applies to immersed submanifolds. We consider embedded for sake of simplicity.

Notice that X need not be tangent to Σ. However, if X is tangent to Σ, its tangential divergence coincides with the intrinsic divergence associated with the induced metric.

Definition 2.2
A *compactly supported variation* of Σ is a smooth map $F\colon \Sigma \times (-\varepsilon, \varepsilon) \to M$ such that $F(x,0) = x$ and $F(x,t) = x$ outside a compact subset of Σ. Its variational field is

$$X(x) = \left.\frac{\partial F(x,t)}{\partial t}\right|_{t=0}.$$

Proposition 2.3 - First variation of volume
The first variation of the k-dimensional volume of Σ is

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}(F_t(\Sigma)) = \int_\Sigma \mathrm{div}_\Sigma X \,\mathrm{d}\,\mathrm{vol}_\Sigma \tag{2.1}$$

$$= -\int_\Sigma \langle \vec{H}, X\rangle \mathrm{d}\,\mathrm{vol}_\Sigma + \int_{\partial\Sigma} \langle X, \eta\rangle \mathrm{d}\,\mathrm{vol}_{\partial\Sigma}, \tag{2.2}$$

where $\vec{H}$ is the (unnormalized) mean curvature vector and η is the outward pointing unit conormal along $\partial\Sigma$.

Notation 2.2
We will write

$$\delta\Sigma(X) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{vol}(F_t(\Sigma)).$$

Proof. Let us start with proving (2.1). Choose local coordinates $x^1, \dots, x^k$ on Σ and write

$$g_{ij}(t) = g(\partial_i F, \partial_j F), \qquad \mathrm{d}\,\mathrm{vol}_{F_t(\Sigma)} = \sqrt{\det g_{ij}(t)}\mathrm{d}x^1 \cdots \mathrm{d}x^k.$$

The ratio of induced volume densities is

$$v(t) := \frac{\mathrm{d}\,\mathrm{vol}_{F_t(\Sigma)}}{\mathrm{d}\,\mathrm{vol}_\Sigma} = \frac{\sqrt{\det g_{ij}(t)}}{\sqrt{\det g_{ij}(0)}}.$$

At a point $p \in \Sigma$ choose coordinates orthonormal at $t = 0$, so $g_{ij}(0) = \delta_{ij}$. Then

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} v(t)\right|_{t=0} = \frac{1}{2}\left.\frac{\mathrm{d}}{\mathrm{d}t} \det g_{ij}(t)\right|_{t=0} = \frac{1}{2}\sum_{i=1}^k \left.\frac{\mathrm{d}}{\mathrm{d}t} g_{ii}(t)\right|_{t=0}$$

$$= \sum_{i=1}^k \langle \nabla_{\partial_t F} \partial_i F, \partial_i F\rangle = \sum_{i=1}^k \langle \nabla_{\partial_i F} \partial_t F, \partial_i F\rangle = \mathrm{div}_\Sigma X,$$

which gives the desired equality.

In order to prove (2.2), decompose $X = X^\perp + X^\top$, which are its normal and tangential components respectively. For an orthonormal tangent frame $\{e_1, \dots, e_k\}$, we have

$$\mathrm{div}_\Sigma X^\perp = \sum_{i=1}^k \langle D_{e_i} X^\perp, e_i\rangle = -\sum_{i=1}^k \langle X^\perp, D_{e_i} e_i\rangle$$

$$= -\sum_{i=1}^k \langle X^\perp, A(e_i, e_i)\rangle = -\langle X^\perp, \vec{H}\rangle = -\langle X, \vec{H}\rangle,$$

where we are using that $\langle X^\top, \vec{H}\rangle = 0$. Since $X^\top$ is tangent to Σ we have

$$\operatorname{div}_\Sigma X^\top = \operatorname{div}_\Sigma^{\text{intrinsic}} X^\top.$$

We get (2.2) by the intrinc divergence theorem, indeed:

$$\begin{aligned}\int_\Sigma \operatorname{div}_\Sigma X \mathrm{d}\operatorname{vol}_\Sigma &= \int_\Sigma \operatorname{div}_\Sigma X^\perp \mathrm{d}\operatorname{vol}_\Sigma + \int_\Sigma \operatorname{div}_\Sigma X^\top \mathrm{d}\operatorname{vol}_\Sigma \\ &= -\int_\Sigma \langle \vec{H}, X\rangle \mathrm{d}\operatorname{vol}_\Sigma + \int_{\partial\Sigma} \langle X, \eta\rangle \mathrm{d}\operatorname{vol}_{\partial\Sigma},\end{aligned}$$

where one uses $\langle X^\perp, \eta\rangle = 0$. □

Definition 2.4
A submanifold $\Sigma \subset M$ is *minimal* if its mean curvature vector $\vec{H}$ vanishes identically. Equivalently, Σ is a critical point of the area functional for every compactly supported variation fixing $\partial\Sigma$.

Remark 2.3
Area minimizers are minimal, but minimal submanifolds need not globally minimize area.

2.2 Examples in $\mathbb{R}^3$

- **Plane:** $\Sigma = \{z = 0\} \subset \mathbb{R}^3$ has $\vec{H} = 0$.
- **Disk:** $\Sigma = \{(x, y, 0) : x^2 + y^2 \le 1\}$ is minimal with boundary. For compactly supported variations and constant vertical displacement $X = (0, 0, 1)$ the first variation is zero; an outward radial variation $X = (x, y, 0)$ changes its area, owing to the boundary term:

$$\delta\Sigma(X) = 2\pi = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \pi(1 + t)^2.$$

- **Catenoid:** A surface of revolution, for instance $x^2 + y^2 = \cosh^2 z$. Two sufficiently separated coaxial circles cannot be joined by an area-minimizing catenoid; compare with two disks. Discovered by Euler (1744) [18]; studied by Meusnier (1776, with a memoir published in 1785) [37]. Recall $\cosh z = (e^z + e^{-z})/2$.
- **Helicoid:** One parametrization is

$$F(t, s) = (t\cos s, t\sin s, s), \qquad s, t \in \mathbb{R}.$$

  Away from the axis its equation can be expressed locally as $z = \arctan(y/x)$, modulo branches. The helicoid is a complete embedded minimal surface, simply connected, and ruled. Meusnier (1776); Catalan (1842) showed it is the only nonplanar ruled minimal surface [6]. For illustrations of the helicoid, see the gallery by Matthias Weber [51].

2.3 The minimal surface equation for graphs

For a graph $\Sigma^n = \{(x, u(x)) : x \in \Omega \subset \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$, the minimal surface equation is

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0.$$

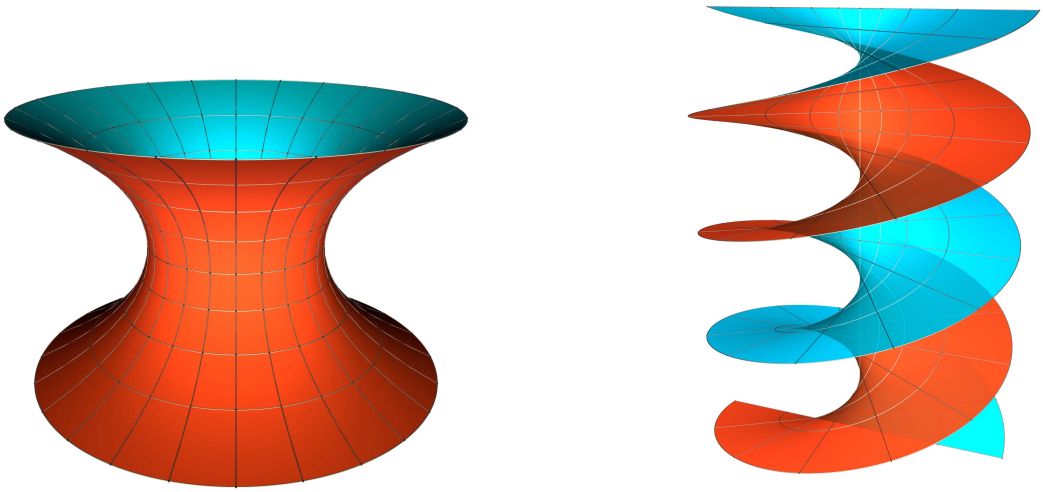


Figure 2.1: Catenoid (left) and helicoid (right) from the minimal surfaces gallery of Matthias Weber [51].

In coordinates this can be written as

$$\Delta u - \frac{\sum_{i,j=1}^{n} u_i u_j u_{ij}}{1+|\nabla u|^2} = 0, \qquad \text{or} \qquad \sum_{i,j=1}^{n} \left(\delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}\right) u_{ij} = 0.$$

This is a second-order quasilinear elliptic PDE.

2.4 First consequences of minimality

Proposition 2.5
A submanifold $\Sigma^k \subset \mathbb{R}^n$ is minimal if and only if the restrictions of the Euclidean coordinate functions $x^1, \dots, x^n$ to Σ are harmonic.

Proof. Let $E_j = \nabla x^j$ in $\mathbb{R}^n$ and $\varphi \in C_c^\infty(\Sigma \setminus \partial\Sigma)$. Set $X = \varphi E_j$. Since the Euclidean connection makes E_j parallel in cap 0 mettere definizione trasporto parallelo,

$$\operatorname{div}_\Sigma X = \sum_{i=1}^{k} \langle \nabla_{e_i}(\varphi E_j), e_i \rangle = \langle \nabla_\Sigma \varphi, E_j \rangle.$$

Therefore

$$\begin{aligned} \int_\Sigma \varphi \Delta_\Sigma x^j \,\mathrm{d}\operatorname{vol}_\Sigma &= -\int_\Sigma \langle \nabla_\Sigma \varphi, \nabla_\Sigma x^j \rangle \,\mathrm{d}\operatorname{vol}_\Sigma \\ &= -\int_\Sigma \operatorname{div}_\Sigma(\varphi E_j) \,\mathrm{d}\operatorname{vol}_\Sigma = \int_\Sigma \varphi \langle \vec{H}, E_j \rangle \,\mathrm{d}\operatorname{vol}_\Sigma . \end{aligned}$$

Thus $\Delta_\Sigma \vec{x} = \vec{H}$ and the claim follows. □

Proposition 2.6
Let $\Sigma^k \subset \mathbb{R}^n$ be a compact minimal submanifold with boundary. Then

$$k \operatorname{area}(\Sigma) = \int_{\partial\Sigma} \langle x, \eta \rangle \,\mathrm{d}\operatorname{vol}_{\partial\Sigma} .$$

If $\Sigma \subset B_R(0)$ and $\partial\Sigma \subset \partial B_R(0)$, then

$$\operatorname{area}(\Sigma) \leq \frac{R}{k} \operatorname{area}(\partial\Sigma).$$

Proof. Take $X(x) = x$. In Euclidean space $\nabla_v X = v$, so $\operatorname{div}_\Sigma X = k$. Apply the first variation/divergence formula and $\vec{H} = 0$. □

Remark 2.4
For a minimal $\Sigma^k \subset \mathbb{R}^n$, $\Delta_\Sigma |x|^2 = 2k$.

Proposition 2.7 - Convex hull property
If $\Sigma^k \subset \mathbb{R}^n$ is a compact minimal submanifold with boundary, then

$$\Sigma \subset \operatorname{Conv}(\partial\Sigma).$$

Proof. For every fixed unit vector v, the function $u(x) = \langle v, x \rangle$ is harmonic on Σ and attains its maximum on $\partial\Sigma$ by the maximum principle. Hence any half-space containing $\partial\Sigma$ contains Σ. Intersecting these half-spaces proves the claim. □

Corollary 2.8
There are no compact minimal submanifolds of $\mathbb{R}^n$ without boundary (of positive dimension).

Corollary 2.9
Let $\Sigma^2 \subset \mathbb{R}^n$ be a compact minimal surface with boundary, homeomorphic to a disk. Let $K \subset \mathbb{R}^n$ be compact and convex with $K \cap \partial\Sigma = \emptyset$. Then $K \cap \Sigma$ is simply connected.

Sketch. By contradiction, suppose that a closed curve $\gamma \subset K \cap \Sigma$ does not bound a disk in $K \cap \Sigma$. Since Σ is disk-like, γ bounds a disk $\Gamma \subset \Sigma$. By the convex hull property, $\Gamma \subset \operatorname{Conv}(\gamma) \subset K$, a contradiction. □

Corollary 2.10
Let $\Sigma^k \subset \mathbb{R}^n$ be a compact minimal submanifold and K compact convex, with $K \cap \partial\Sigma = \emptyset$. Then the inclusion $\Sigma \cap K \hookrightarrow \Sigma$ induces an injection on the $(k-1)$-st homology group.

2.5 Monotonicity formula

Let $\Sigma^k \subset \mathbb{R}^n$ be a minimal submanifold and set $\omega_k = |B_1^k(0)|$. We denote by $B_r(x_0)$ the Euclidean ball in $\mathbb{R}^n$, and by $B_r^\Sigma(o)$ the intrinsic geodesic ball in Σ.

Theorem 2.11 - Monotonicity formula(i) *(Extrinsic version) For $x_0 \in \mathbb{R}^n$ and $0 < s < r < \operatorname{dist}(x_0, \partial\Sigma)$, assuming the intersections have finite volume, we have*

$$\frac{|\Sigma \cap B_r(x_0)|}{r^k} - \frac{|\Sigma \cap B_s(x_0)|}{s^k} = \int_{\Sigma \cap (B_r(x_0) \setminus B_s(x_0))} \frac{|(x - x_0)^\perp|^2}{|x - x_0|^{k+2}} \, \mathrm{d}\operatorname{vol}_\Sigma . \tag{2.3}$$

In particular,

$$\Theta_\Sigma^{\mathrm{ext}}(x_0, r) := \frac{|\Sigma \cap B_r(x_0)|}{\omega_k r^k}$$

is monotone nondecreasing. It is constant in r if and only if Σ is conical with vertex x_0 (in the relevant annulus).

If $x_0 \in \Sigma$ is a regular point, then $\Theta^{\mathrm{ext}}_{\Sigma}(x_0, r) \geq 1$; equality for some r holds if and only if $\Sigma \cap B_r(x_0)$ is a flat k-disk, under the usual connectedness and no-boundary assumptions.

(ii) *(Intrinsic version) For $x_0 \in \Sigma$ and $0 < s < r < \operatorname{dist}_{\Sigma}(x_0, \partial\Sigma)$, write $r_{\Sigma}(x) = \operatorname{dist}_{\Sigma}(x_0, x)$ and choose Euclidean coordinates so that x_0 is mapped to 0. Then*

$$\frac{|B^{\Sigma}_r(x_0)|}{r^k} - \frac{|B^{\Sigma}_s(x_0)|}{s^k} = \int_{B^{\Sigma}_r(x_0)\setminus B^{\Sigma}_s(x_0)} \frac{1}{r_{\Sigma}(x)^{k+1}} \Bigg[\big(r_{\Sigma}(x) - |x|\big) + \frac{|x - x_0|}{2} \left| \frac{x - x_0}{|x - x_0|} - \nu_{B^{\Sigma}_{r_{\Sigma}(x)}(x_0)}(x) \right|^2 \Bigg] \mathrm{d}\,\mathrm{vol}_{\Sigma}(x). \tag{2.4}$$

Here $\nu_{B^{\Sigma}_t(x_0)}$ is the outward intrinsic unit normal, defined almost everywhere, and the integrand at $x = 0$ is interpreted by continuity. Thus

$$\Theta^{\mathrm{int}}_{\Sigma}(x_0, r) := \frac{|B^{\Sigma}_r(x_0)|}{\omega_k r^k}$$

is also monotone nondecreasing, with $\Theta^{\mathrm{int}}_{\Sigma}(x_0, r) \geq 1$. The intrinsic formula holds with the standard almost-everywhere interpretation at the cut locus [24].

Proof. For the extrinsic formula, we can assume $x_0 = 0$ and that Σ intersects ∂B_s and ∂B_r transversally. Consider

$$X(x) = \frac{x}{|x|^k}, \qquad DX = \frac{\mathrm{Id}}{|x|^k} - \frac{k\, x \otimes x^{\flat}}{|x|^{k+2}}.$$

If $\{e_i\}_{i=1}^k$ is an orthonormal basis of $T_x\Sigma$, then

$$\operatorname{div}_{\Sigma} X = \sum_{i=1}^{k} \langle D_{e_i} X, e_i \rangle = \frac{k}{|x|^k} - \frac{k|x^{\top}|^2}{|x|^{k+2}} = \frac{k|x^{\perp}|^2}{|x|^{k+2}}.$$

Integrating on $\Sigma \cap (B_r \setminus B_s)$ and applying the divergence theorem, together with $\operatorname{div}_{\Sigma} x = k$, gives (2.3). The general result follows by approximation by regular radii. If the density is constant, $x^{\perp} = 0$ in the annulus, hence the radial vector field is tangent to Σ and Σ is conical. Finally, if $x_0 \in \Sigma$ is regular, $\lim_{r\to 0} \Theta^{\mathrm{ext}}_{\Sigma}(x_0, r) = 1$.

For the intrinsic formula, assume again that $x_0 = 0$. Recall that $\Delta_{\Sigma}|x|^2 = 2k$, so the divergence theorem on a regular intrinsic ball yields

$$k|B^{\Sigma}_R| = \frac{1}{2}\int_{B^{\Sigma}_R} \Delta_{\Sigma}|x|^2 \,\mathrm{d}\,\mathrm{vol}_{\Sigma} = \int_{\partial B^{\Sigma}_R} \langle x, \nu_{B^{\Sigma}_R} \rangle \,\mathrm{d}\,\mathrm{vol}_{\partial B^{\Sigma}_R}.$$

Since

$$\langle x, \nu_{B^{\Sigma}_R} \rangle = R - \left[(R - |x|) + \frac{|x|}{2} \left| \frac{x}{|x|} - \nu_{B^{\Sigma}_R} \right|^2 \right],$$

we get

$$R|\partial B^{\Sigma}_R| - k|B^{\Sigma}_R| = \int_{\partial B^{\Sigma}_R} \left[(R - |x|) + \frac{|x|}{2} \left| \frac{x}{|x|} - \nu_{B^{\Sigma}_R} \right|^2 \right] \mathrm{d}\,\mathrm{vol}_{\partial B^{\Sigma}_R}.$$

Now use

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{|B_R^\Sigma|}{R^k}\right) = \frac{R|\partial B_R^\Sigma| - k|B_R^\Sigma|}{R^{k+1}}$$

and the coarea formula to obtain (2.4). Since $r_\Sigma(x) \geq |x|$, its right-hand side is nonnegative. □

Remark 2.5
The monotonicity formulas also hold for immersed submanifolds, with volume counted with multiplicity. In particular, $\lim_{r\to 0} \Theta_\Sigma^{\mathrm{ext}}(x_0, r)$ is the multiplicity at x_0 (for a smooth immersion). The extrinsic formula extends to weaker notions of minimal submanifolds and is fundamental in regularity theory.

Corollary 2.12 - Florit–Simon [24]
Let $\Sigma^k \subset \mathbb{R}^n$ be a complete, connected, smoothly immersed minimal submanifold, without boundary, and $o \in \Sigma$. Then

$$\lim_{R\to\infty} \Theta_\Sigma^{\mathrm{int}}(o, R) = \lim_{R\to\infty} \Theta_\Sigma^{\mathrm{ext}}(o, R) \in [1, +\infty].$$

In particular, bounded intrinsic area density is equivalent to bounded extrinsic area density, and either bound implies that the immersion is proper.

Remark 2.6
The properness assumption cannot be omitted in statements about bounded complete minimal surfaces. Jorge–Xavier constructed a complete minimal immersion in $\mathbb{R}^3$ between two parallel planes [32], and Nadirashvili constructed a complete minimal immersion in $\mathbb{R}^3$ contained in a ball [41]. Both are nonproper.

Theorem 2.13 - Extended monotonicity
Let $\Sigma^k \subset \mathbb{R}^n$ be a compact minimal submanifold with smooth boundary and $x_0 \in \mathbb{R}^n \backslash \partial\Sigma$. Define the exterior cone over $\partial\Sigma$ with vertex x_0 by

$$E = E(x_0, \partial\Sigma) := \{x_0 + t(x - x_0) : t \geq 1,\ x \in \partial\Sigma\}, \qquad \widetilde{\Sigma} = \Sigma \cup E.$$

The volume of E is counted with multiplicity when needed. Then

$$\Theta_{\widetilde{\Sigma}}^{\mathrm{ext}}(x_0, r) := \frac{|\widetilde{\Sigma} \cap B_r(x_0)|}{\omega_k r^k}$$

is monotone nondecreasing for $r > 0$. Equality on an interval of radii forces the radial vector field to be tangent to $\widetilde{\Sigma}$ in the corresponding annulus, with vanishing boundary contribution along $\partial\Sigma$.

Proof. On the cone E, the position vector $x - x_0$ is tangent, so $\langle x - x_0, \vec{H}_E\rangle = 0$. On $\partial\Sigma$, the outward conormals of Σ and E satisfy

$$\langle x - x_0, \eta_\Sigma + \eta_E\rangle \leq 0.$$

Using the divergence theorem for $X = (x - x_0)/|x - x_0|^k$ on both pieces, the common boundary gives a nonnegative contribution to the variation of the density. The remaining term is the same nonnegative integrand as in (2.3). □

Theorem 2.14 - Fáry–Milnor [19, 38]
Let $\Gamma \subset \mathbb{R}^3$ be a smooth simple closed curve. If

$$\int_\Gamma |\vec{k}| \, \mathrm{d}s \leq 4\pi,$$

where $\vec{k}$ is the curvature vector, then Γ is unknotted.

Sketch. First assume $\int_\Gamma |\vec{k}| \, \mathrm{d}s < 4\pi$. Let Σ be an area-minimizing disk with boundary Γ, provided by the Douglas–Radó solution to the Plateau problem [17, 46]. Interior regularity gives a smooth minimal immersion (and in this setting no interior branch points). It is enough to prove that Σ is embedded.

Suppose by contradiction that two sheets of Σ meet at an interior point $x_0 \in \Sigma$. Then

$$\lim_{r \to 0} \Theta^{\mathrm{ext}}_{\Sigma}(x_0, r) \geq 2.$$

Let $E = E(x_0, \Gamma)$ and $\widetilde{\Sigma} = \Sigma \cup E$. By extended monotonicity,

$$2 \leq \lim_{r \to 0} \Theta^{\mathrm{ext}}_{\widetilde{\Sigma}}(x_0, r) \leq \lim_{R \to \infty} \Theta^{\mathrm{ext}}_{\widetilde{\Sigma}}(x_0, R). \tag{2.5}$$

On the other hand, the compact part Σ does not contribute to the density at infinity. By the conical structure,

$$\lim_{R \to \infty} \frac{|E \cap B_R(x_0)|}{\pi R^2} = \lim_{R \to \infty} \frac{|E \cap \partial B_R(x_0)|}{2\pi R}.$$

The annulus $E \cap B_R(x_0)$ is flat away from the vertex. By Gauss–Bonnet, its two boundary components have opposite total geodesic curvature. The outer boundary contributes its length divided by R, and the contribution at Γ is bounded by the total curvature of Γ:

$$\frac{|E \cap \partial B_R(x_0)|}{R} = -\int_\Gamma \langle \vec{k}, \eta_E \rangle \, \mathrm{d}s \leq \int_\Gamma |\vec{k}| \, \mathrm{d}s.$$

Therefore

$$\lim_{R \to \infty} \Theta^{\mathrm{ext}}_{\widetilde{\Sigma}}(x_0, R) \leq \frac{1}{2\pi} \int_\Gamma |\vec{k}| \, \mathrm{d}s < 2,$$

contradicting (2.5). The equality case $\int_\Gamma |\vec{k}| \, \mathrm{d}s = 4\pi$ follows from the rigidity argument in the Fáry–Milnor theorem [19, 38]. □

2.6 The halfspace theorem

Theorem 2.15 - Hoffman–Meeks halfspace theorem [27]
Let $\Sigma^2 \subset \mathbb{R}^3$ be a connected, complete, properly embedded minimal surface. If Σ is contained in a halfspace, then Σ is a plane.

To prove this result, we need a version of the maximum principle for minimal hypersurfaces.

Theorem 2.16 - Strong maximum principle
If $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ are connected minimal hypersurfaces, tangent at an interior point, and one lies locally on one side of the other, then they agree locally. Consequently, if $U \subset \mathbb{R}^{n+1}$ is open, $\Sigma_1 \cap U$ and $\Sigma_2 \cap U$ are connected, $\Sigma_1 \cap U = \partial\Omega$ for an open set

$\Omega \subset U$, and $\Sigma_2 \cap U \subset \overline{\Omega}$, then either the two pieces are disjoint or they agree.

Sketch of the halfspace theorem. We use the family of half-catenoids as barriers. Up to a rigid motion, assume $\Sigma \subset \{z \geq 0\}$ and that $\inf_\Sigma z = 0$. If Σ touches $\{z = 0\}$, the maximum principle implies that it is a plane. Thus we may assume that $\Sigma \cap \{z = 0\} = \varnothing$.

For $r > 0$ and $\varepsilon > 0$, consider the lower half of a catenoid with neck of radius r at height ε:

$$\Gamma_{\varepsilon,r} = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{c} x^2 + y^2 = r^2 \cosh^2\left(\dfrac{z-\varepsilon}{r}\right), \\ z \leq \varepsilon \end{array} \right\}.$$

Its boundary is the circle $\{x^2 + y^2 = r^2,\ z = \varepsilon\}$. Properness allows us to choose $\varepsilon > 0$ sufficiently small so that Σ is disjoint from $\Gamma_{\varepsilon,1}$ and from its boundary. As $r \downarrow 0$, $\Gamma_{\varepsilon,r}$ converges, away from the vertical axis, to the punctured plane $\{z = \varepsilon\}$, which meets Σ since $\inf_\Sigma z = 0$.

We now vary r between 0 and 1. For $0 < r \leq 1$, the boundaries $\partial\Gamma_{\varepsilon,r}$ remain disjoint from Σ. Moreover, any intersections lie in a fixed compact region: far from the axis the half-catenoids lie below $\{z = 0\}$. At a first value of r for which the two surfaces cease to intersect, there is therefore an interior tangency. This contradicts the maximum principle. The complete argument, including the barrier construction, is in [27]. □

Remark 2.7
The hypothesis of properness is essential. Jorge–Xavier constructed a complete immersed minimal surface between two parallel planes [32].

Remark 2.8
Complete embedded minimal surfaces of finite topology in $\mathbb{R}^3$ are proper. This is a consequence of the work of Colding–Minicozzi [12]. It remains open whether every complete embedded minimal surface of finite genus is proper. Properness is known when the set of ends is countable [35].

The Riemann minimal examples, introduced around 1860, are complete, properly embedded, singly periodic minimal surfaces of genus zero with infinitely many planar ends. They provide examples of much more complicated global geometry than planes, catenoids, and helicoids.

2.7 The Weierstrass representation

We now consider minimal surfaces Σ^2 in $\mathbb{R}^3$. Complex geometry provides a powerful method for producing examples.

Let $F\colon \Sigma \to \mathbb{R}^3$ be smooth. Recall that F is an *immersion* if dF_p is injective for every $p \in \Sigma$, a *branched immersion* if this fails only at isolated points, and an *embedding* if F is an immersion and a homeomorphism onto its image.

Theorem 2.17 - Isothermal coordinates
Every point of a smooth Riemannian surface (Σ^2, g) admits local coordinates (x, y) such that

$$g = \mu^2(dx^2 + dy^2), \qquad \mu > 0.$$

Moreover, x and y are harmonic with respect to g.

Set $z = x + iy$, $\overline{z} = x - iy$, and

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Since $\Delta_\Sigma = \mu^{-2}(\partial_x^2 + \partial_y^2)$ on functions, a map $F\colon \Sigma \to \mathbb{R}^3$ is harmonic if and only if $\partial_{\overline{z}}\partial_z F = 0$.

Suppose that F is an isothermal immersion and define

$$\Phi = 2\partial_z F = F_x - iF_y = (\phi_1, \phi_2, \phi_3).$$

Conformality is equivalent to

$$\Phi \cdot \Phi = \phi_1^2 + \phi_2^2 + \phi_3^2 = 0, \qquad |\Phi|^2 = 2\mu^2,$$

where the first dot product is complex bilinear and the second norm is Hermitian. Moreover,

$$F \text{ is minimal} \iff \Delta_\Sigma F = 0 \iff \partial_{\overline{z}}\Phi = 0.$$

Thus the components of Φ are holomorphic. Also,

$$dF = \operatorname{Re}(\Phi\, dz), \qquad F(p) = F(p_0) + \operatorname{Re}\int_{p_0}^{p} \Phi\, dz.$$

If Σ is not simply connected, the integral is independent of path precisely when all real periods vanish.

Assume that F is oriented, and let $N\colon \Sigma \to S^2$ be its unit normal, with the orientation induced by F. Denote by $\pi\colon S^2 \to \mathbb{C} \cup \{\infty\}$ stereographic projection from the north pole. The map $g := \pi \circ N$ is the *meromorphic Gauss map*. In terms of Φ it is given by

$$g = \frac{\phi_3}{\phi_1 - i\phi_2} = -\frac{\phi_1 + i\phi_2}{\phi_3},$$

whenever the quotients make sense. It follows that

$$\phi_1 = \frac{1}{2}\left(g^{-1} - g\right)\phi_3, \qquad \phi_2 = \frac{i}{2}\left(g^{-1} + g\right)\phi_3.$$

Writing $dh = \phi_3\, dz$, we obtain the Weierstrass representation

$$\boxed{F = \operatorname{Re}\int \left(\frac{1}{2}\left(g^{-1} - g\right), \frac{i}{2}\left(g^{-1} + g\right), 1\right) dh.}$$

Here dh is a holomorphic 1-form, and the apparent singularities in the expression must cancel so that the vector-valued 1-form is holomorphic. The induced metric is

$$ds^2 = \frac{1}{4}\left(|g| + |g|^{-1}\right)^2 |dh|^2.$$

At zeros and poles of g the expression is interpreted by continuity.

Proposition 2.18 - Weierstrass representation
Let Σ be a Riemann surface, g a meromorphic function, and dh a holomorphic 1-form such that

$$\Phi\, dz = \left(\frac{1}{2}(g^{-1} - g), \frac{i}{2}(g^{-1} + g), 1\right) dh$$

is holomorphic and has no common zeros in its three components. If all its periods are purely imaginary, then

$$F(p) = \operatorname{Re}\int_{p_0}^{p} \Phi\, dz$$

defines a conformal minimal immersion $F\colon \Sigma \to \mathbb{R}^3$. If the components have isolated

common zeros, the same formula defines a branched minimal immersion. On a simply connected Σ, the period condition is automatic.

The Gauss map can be recovered from g by

$$N = \frac{1}{1+|g|^2}(2\operatorname{Re} g, 2\operatorname{Im} g, |g|^2 - 1).$$

Indeed, the right-hand side has norm one and is orthogonal to both $\operatorname{Re}\Phi$ and $\operatorname{Im}\Phi$.

Example 2.9 - Classical Weierstrass data (i) The *catenoid*: $\Sigma = \mathbb{C}\setminus\{0\}$,

$$g(z) = z, \qquad dh = \frac{dz}{z}.$$

(ii) The *helicoid*: $\Sigma = \mathbb{C}$,

$$g(z) = e^{iz}, \qquad dh = dz.$$

(iii) *Enneper's surface* (1864): $\Sigma = \mathbb{C}$,

$$g(z) = z, \qquad dh = z\,dz.$$

Up to a translation and a dilation, a parametrization in real coordinates $z = s + it$ is

$$F(s,t) = \left(s - \frac{s^3}{3} + st^2, -t + \frac{t^3}{3} - s^2 t,\ s^2 - t^2\right).$$

The Weierstrass representation is especially useful for constructing complete minimal surfaces. However, checking the period condition, completeness, and embeddedness can be difficult.

For instance, Costa [14] constructed a complete minimal immersion in $\mathbb{R}^3$ of genus one with three embedded ends, using elliptic functions. Hoffman–Meeks [28] later proved that this surface is embedded; see Figure 2.2. For many years, the plane, catenoid, and helicoid were the only known complete embedded minimal surfaces of finite topology in $\mathbb{R}^3$.

2.8 The Bernstein theorem

Theorem 2.19 - Bernstein [4]
Let $u\colon \mathbb{R}^2 \to \mathbb{R}$ be an entire smooth solution of the minimal surface equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0.$$

Then $u(x,y) = ax + by + c$ for some constants $a, b, c \in \mathbb{R}$. Equivalently, an entire minimal graph in $\mathbb{R}^3$ is a plane.

We will use the following two lemmas.

Lemma 2.20 - Curvature estimate
Let $\Sigma^2 \subset \mathbb{R}^3$ be an entire minimal graph. There exists a constant $C > 0$ such that, for every compactly supported Lipschitz function η on Σ,

$$\int_\Sigma \eta^2 |A|^2 \,\mathrm{d}\mu \le C \int_\Sigma |\nabla^\Sigma \eta|^2 \,\mathrm{d}\mu.$$

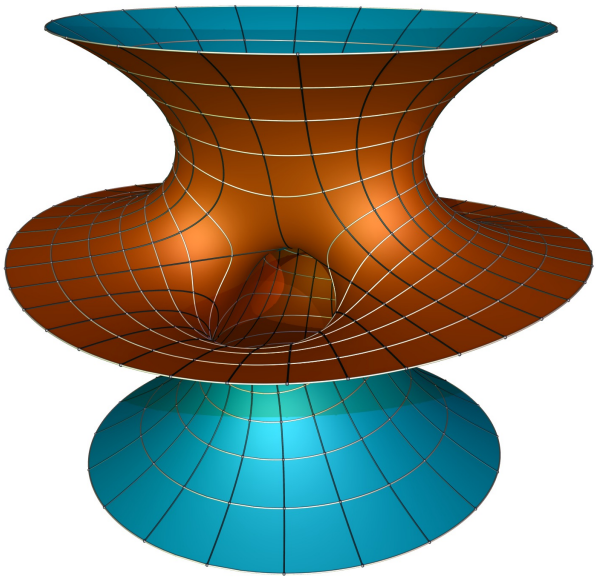


Figure 2.2: Costa surface. Image from Weber's minimal surfaces repository [51].

Proof. Let $N\colon \Sigma \to S^2$ be the upward pointing Gauss map. Since Σ is a graph, its image lies in the open upper hemisphere $S^2_+ = \{v_3 > 0\}$. There exists a smooth 1-form α on the closed upper hemisphere with

$$d\alpha = \mathrm{d}\,\mathrm{vol}_{S^2}, \qquad |\alpha| \le C_0.$$

Since Σ is minimal, $H = 0$ and its principal curvatures satisfy $\lambda_1 = -\lambda_2$. Consequently,

$$|A|^2 = \lambda_1^2 + \lambda_2^2 = -2\lambda_1\lambda_2 = -2K, \qquad N^*(\mathrm{d}\,\mathrm{vol}_{S^2}) = K\,\mathrm{d}\mu.$$

In particular,

$$\begin{aligned}\int_\Sigma \eta^2 |A|^2\,\mathrm{d}\mu &= -2\int_\Sigma \eta^2\, d(N^*\alpha)\\ &= 4\int_\Sigma \eta\, d\eta \wedge N^*\alpha\\ &\le 4C_0\int_\Sigma \eta|\nabla^\Sigma \eta||A|\,\mathrm{d}\mu.\end{aligned}$$

The conclusion follows from Cauchy–Schwarz (and approximation for Lipschitz cutoffs). □

Lemma 2.21 - Quadratic area bound
Let $\Sigma^2 \subset \mathbb{R}^3$ be an entire minimal graph. Then for every $r > 0$ and every $x_0 \in \mathbb{R}^3$,

$$|\Sigma \cap B_r(x_0)| \le 2\pi r^2.$$

Proof. Extend the upward unit normal N to $\mathbb{R}^3$ by keeping it constant along vertical lines. Consider the 2-form

$$\omega(X, Y) = \det(X, Y, N).$$

Since u solves the minimal surface equation, $\operatorname{div} N = 0$, and hence $d\omega = 0$. Moreover, $|\omega(X,Y)| \le 1$ for orthonormal X, Y, with equality on the oriented tangent planes of Σ. Thus ω calibrates Σ, and Σ is area-minimizing.

For almost every r, $\Sigma \cap \partial B_r(x_0)$ is a smooth union of closed curves. The graph separates $B_r(x_0)$ into an upper and a lower region; the corresponding pieces of $\partial B_r(x_0)$ have areas adding up to $4\pi r^2$. By the calibration and Stokes' theorem, $\Sigma \cap B_r(x_0)$ has area no larger than either of these two spherical pieces. Therefore

$$|\Sigma \cap B_r(x_0)| \le \frac{1}{2}|\partial B_r(x_0)| = 2\pi r^2.$$

The conclusion for every r follows by approximation. □

Proof of Bernstein's theorem. Fix $R > 2$ and define the radial cutoff $\eta_R \colon \mathbb{R}^3 \to [0,1]$ by

$$\eta_R(x) = \begin{cases} 1, & |x| \le R, \\ 2 - \dfrac{\log|x|}{\log R}, & R < |x| < R^2, \\ 0, & |x| \ge R^2. \end{cases}$$

Since $|\nabla^\Sigma |x|| \le 1$, we have

$$|\nabla^\Sigma \eta_R| \le \frac{1}{|x| \log R} \quad \text{on } B_{R^2} \setminus B_R.$$

By the curvature estimate and the quadratic area bound,

$$\begin{aligned} \int_{\Sigma \cap B_R} |A|^2 \, \mathrm{d}\mu &\le \int_\Sigma \eta_R^2 |A|^2 \, \mathrm{d}\mu \\ &\le \frac{C}{(\log R)^2} \int_{\Sigma \cap (B_{R^2} \setminus B_R)} \frac{1}{|x|^2} \, \mathrm{d}\mu \\ &\le \frac{C'}{\log R}. \end{aligned}$$

For the last inequality, decompose the annulus into dyadic annuli, or integrate by parts the area function $A(r) = |\Sigma \cap B_r|$, using $A(r) \le 2\pi r^2$. Letting $R \to \infty$ gives $|A| \equiv 0$. Thus Σ is totally geodesic and therefore a plane. □

Remark 2.10 - The Bernstein problem in higher dimensions
A natural question is whether an entire minimal graph $\Sigma^n \subset \mathbb{R}^{n+1}$ must be a hyperplane. The answer is positive for $n \le 7$, by results of Fleming [23], De Giorgi [15], Almgren [1], and Simons [50].

In contrast, Bombieri–De Giorgi–Giusti [5] constructed non-affine entire solutions for $n \ge 8$. The related Simons cone in $\mathbb{R}^8$ is

$$\mathcal{C} = \{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\},$$

a singular area-minimizing hypersurface. Thus the dimension threshold for the Bernstein problem is sharp.

2.9 Second variation formula and Morse index

Proposition 2.22 - Second variation formula
Let $F\colon \Sigma^k \times (-\varepsilon,\varepsilon) \to M^n$ be a variation with compact support in the interior of a minimal submanifold $\Sigma^k \subset M^n$. Assume that the variational field

$$X = \left.\frac{\partial F}{\partial t}\right|_{t=0}$$

is normal to Σ. Then

$$\begin{aligned}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \mathrm{vol}(F(\Sigma,t)) &= \int_\Sigma \left(|\nabla^\perp X|^2 - \sum_{i,j=1}^k \langle A(e_i,e_j),X\rangle^2 - \sum_{i=1}^k R^M(X,e_i,X,e_i)\right)\mathrm{d}\,\mathrm{vol}_\Sigma \\ &=: Q^\Sigma(X,X),\end{aligned}$$

where $\{e_i\}_{i=1}^k$ is a local orthonormal frame of $T\Sigma$ and $\nabla^\perp$ is the normal connection.

Proof. Write $\Sigma_t = F(\Sigma,t)$ and $X_t = \partial_t F$. Recall the first variation formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{vol}(\Sigma_t) = \int_{\Sigma_t} \mathrm{div}_{\Sigma_t} X_t \,\mathrm{d}\,\mathrm{vol}_{\Sigma_t}.$$

In particular, since Σ is minimal and X is normal,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{d}\,\mathrm{vol}_{\Sigma_t} = \mathrm{div}_\Sigma X \,\mathrm{d}\,\mathrm{vol}_\Sigma = -\langle \vec{H}, X\rangle \mathrm{d}\,\mathrm{vol}_\Sigma = 0.$$

Fix local coordinates on Σ, and denote by $e_i(t) = \partial_i F(\cdot,t)$ the corresponding tangent fields along Σ_t. We have

$$g_{ij}(t) = \langle e_i(t), e_j(t)\rangle, \qquad \mathrm{div}_{\Sigma_t} X_t = g^{ij}(t)\langle D_{e_i(t)} X_t, e_j(t)\rangle.$$

At $t=0$, using that X is normal and Σ is minimal,

$$\partial_t g_{ij}|_{t=0} = -2\langle X, A(e_i,e_j)\rangle, \qquad \left.\partial_t g^{ij}\right|_{t=0} = 2\langle X, A(e_i,e_j)\rangle$$

in an orthonormal frame. Moreover, by the curvature identity,

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle D_{e_i(t)} X_t, e_j(t)\rangle &= \langle D_{e_i}(D_t X_t), e_j\rangle + \langle R^M(X,e_i)X, e_j\rangle \\ &\quad + \langle D_{e_i} X, D_{e_j} X\rangle.\end{aligned}$$

The first term, after taking the trace, integrates to zero, since $D_t X_t$ is compactly supported and $\vec{H} = 0$. Also,

$$\sum_i |D_{e_i} X|^2 = |\nabla^\perp X|^2 + \sum_{i,j} \langle A(e_i,e_j),X\rangle^2,$$

while

$$\sum_{i,j} \left.\partial_t g^{ij}\right|_{t=0} \langle D_{e_i} X, e_j\rangle = -2\sum_{i,j}\langle A(e_i,e_j),X\rangle^2.$$

Combining these identities and integrating, we obtain the formula. □

Integrating by parts, we can also write

$$Q^\Sigma(X,X) = -\int_\Sigma \langle X, L_\Sigma X\rangle \,\mathrm{d}\,\mathrm{vol}_\Sigma,$$

where L_Σ is the self-adjoint *Jacobi operator* on normal fields,

$$L_\Sigma X = \Delta_\Sigma^\perp X + \sum_{i=1}^{k} \left(R^M(X, e_i)e_i\right)^\perp + \mathcal{A}(X),$$

with

$$\Delta_\Sigma^\perp X = \sum_{i=1}^{k} \left(\nabla_{e_i}^\perp \nabla_{e_i}^\perp X - \nabla_{\nabla_{e_i}^\Sigma e_i}^\perp X\right), \qquad \mathcal{A}(X) = \sum_{i,j=1}^{k} \langle A(e_i, e_j), X\rangle A(e_i, e_j).$$

The operator $\mathcal{A}$ is also called the *Simons operator*.

Definition 2.23
A minimal submanifold Σ is *stable* if

$$Q^\Sigma(X, X) \geq 0$$

for every compactly supported normal field X, equivalently, for every compactly supported variation.

Definition 2.24
The *Morse index* of a minimal submanifold Σ is the maximal dimension of a linear subspace of $\Gamma_c(N\Sigma)$ on which Q^Σ is negative definite. In particular, Σ is stable if and only if its Morse index is zero.

Let $\Sigma^n \subset M^{n+1}$ be a minimal hypersurface. Since the normal bundle is one-dimensional, the second variation becomes

$$Q^\Sigma(X, X) = \int_\Sigma \left(|\nabla^\perp X|^2 - |A|^2|X|^2 - \mathrm{Ric}^M(X, X)\right) \mathrm{d}\,\mathrm{vol}_\Sigma\,.$$

If Σ is two-sided, let ν be a global unit normal. Every normal field is of the form $X = u\nu$, with $u \in C_c^\infty(\Sigma)$, and

$$\begin{aligned} Q^\Sigma(u, u) &= \int_\Sigma \left(|\nabla u|^2 - (\,\mathrm{Ric}^M(\nu, \nu) + |A|^2)u^2\right) \mathrm{d}\,\mathrm{vol}_\Sigma \\ &= -\int_\Sigma u L_\Sigma u \,\mathrm{d}\,\mathrm{vol}_\Sigma, \end{aligned}$$

where

$$L_\Sigma u = \Delta_\Sigma u + (\,\mathrm{Ric}^M(\nu, \nu) + |A|^2)u.$$

For a minimal surface $\Sigma^2 \subset \mathbb{R}^3$ we simply have

$$Q^\Sigma(u, u) = \int_\Sigma \left(|\nabla u|^2 - |A|^2 u^2\right) \mathrm{d}\,\mathrm{vol}_\Sigma\,.$$

Minimal graphs are area-minimizing, hence stable. Consequently,

$$\int_\Sigma |A|^2 u^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma \leq \int_\Sigma |\nabla u|^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma \qquad \forall u \in C_c^\infty(\Sigma).$$

This resembles the estimate used in the proof of Bernstein's theorem.

2.10 The stable Bernstein theorem

It is natural to ask whether complete, two-sided, stable minimal hypersurfaces in Euclidean space must be planes. This is true up to ambient dimension 7; the last case was established in a recent preprint [29].

Theorem 2.25 - Stable Bernstein theorem
Let $\Sigma^n \to \mathbb{R}^{n+1}$ be a complete, connected, two-sided, stable minimal immersion, with $2 \le n \le 6$. Then its image is an affine hyperplane.

For $n = 2$, this was proved independently by do Carmo–Peng [16], Fischer-Colbrie–Schoen [22], and Pogorelov [45]. We present a proof in this case.

Lemma 2.26 - Intrinsic area bound
Let $\Sigma^2 \to \mathbb{R}^3$ be a complete, two-sided, stable minimal immersion. Then, for every $p \in \Sigma$ and $r > 0$,

$$|B_r^\Sigma(p)| \le \frac{4\pi}{3} r^2.$$

Proof. It is enough to argue on the universal cover, which is again stable, and hence we may assume that Σ is simply connected. Since Σ is minimal,

$$K = -\frac{1}{2}|A|^2 \le 0.$$

Thus, by the Cartan–Hadamard theorem, the intrinsic distance $\rho(x) = d_\Sigma(p, x)$ is smooth away from p, and $B_r^\Sigma(p)$ is a disk.

Set

$$a(r) = |B_r^\Sigma(p)|, \qquad \ell(r) = |\partial B_r^\Sigma(p)|.$$

By the first variation of length and Gauss–Bonnet,

$$\ell'(r) = \int_{\partial B_r^\Sigma(p)} \kappa_g = 2\pi - \int_{B_r^\Sigma(p)} K.$$

Since $a'(r) = \ell(r)$ and $a(0) = \ell(0) = 0$, integrating twice gives

$$\begin{aligned} a(r) &= \pi r^2 - \frac{1}{2}\int_{B_r^\Sigma(p)} K(r-\rho)^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma \\ &= \pi r^2 + \frac{1}{4}\int_{B_r^\Sigma(p)} |A|^2 (r-\rho)^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma\,. \end{aligned}$$

Using stability with $u = (r-\rho)_+$, by approximation,

$$\int_{B_r^\Sigma(p)} |A|^2 (r-\rho)^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma \le \int_{B_r^\Sigma(p)} |\nabla \rho|^2 \,\mathrm{d}\,\mathrm{vol}_\Sigma = a(r).$$

Therefore

$$a(r) \le \pi r^2 + \frac{1}{4} a(r),$$

which proves the claim. □

Remark 2.11
By [24], for a complete, connected minimal immersion in Euclidean space, the intrinsic and extrinsic area densities have the same limit at infinity. In particular, the preceding intrinsic quadratic area bound also gives quadratic extrinsic area growth (counting multiplicity). We do not need this result in the proof below, which uses intrinsic balls throughout.

Proof of the stable Bernstein theorem in $\mathbb{R}^3$. The argument is the same logarithmic-cutoff argument used in the proof of the graphical Bernstein theorem (Theorem 2.19). The difference is that here we use intrinsic balls, with the quadratic area bound given by the preceding lemma.

Fix $p \in \Sigma$ and let $\rho(x) = d_\Sigma(p, x)$. For $R > 1$, set

$$\eta_R(x) = \begin{cases} 1, & \rho(x) \le 1, \\ 1 - \dfrac{\log \rho(x)}{\log R}, & 1 < \rho(x) < R, \\ 0, & \rho(x) \ge R. \end{cases}$$

Set $a(r) = |B_r^\Sigma(p)|$. Since $|\nabla \rho| = 1$ almost everywhere, by stability and the coarea formula,

$$\int_{B_1^\Sigma(p)} |A|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma \le \int_\Sigma |\nabla \eta_R|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma = \frac{1}{(\log R)^2} \int_1^R \frac{a'(s)}{s^2} \, \mathrm{d}s.$$

As in the graphical case, integrating by parts and using $a(s) \le \frac{4\pi}{3} s^2$ gives

$$\int_1^R \frac{a'(s)}{s^2} \, \mathrm{d}s = \frac{a(R)}{R^2} - a(1) + 2 \int_1^R \frac{a(s)}{s^3} \, \mathrm{d}s \le C(1 + \log R).$$

Thus

$$\int_{B_1^\Sigma(p)} |A|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma \le \frac{C}{\log R} \longrightarrow 0.$$

Since p was arbitrary, $A \equiv 0$, and Σ is a plane. □

Remark 2.12 - Higher dimensions
The stable Bernstein theorem in higher dimensions was developed through curvature estimates and rigidity results. Schoen–Simon–Yau [47] obtained the foundational estimates under area-growth assumptions, extended in dimension 6 by Bellettini [3]. Without a volume-growth assumption, the problem in $\mathbb{R}^4$ was solved by Chodosh–Li [9], with another proof by Catino–Mastrolia–Roncoroni [7]; the cases $\mathbb{R}^5$ and $\mathbb{R}^6$ were established by Chodosh–Li–Minter–Stryker [10] and Mazet [34], respectively. The case $\mathbb{R}^7$ was addressed by Hong–Li–Wang [29] (2026 preprint).

The theorem is false in $\mathbb{R}^8$: the Simons cone is area-minimizing, and its Hardt–Simon leaves give complete, non-flat, smooth, stable minimal hypersurfaces [26].

2.11 Compactness and curvature estimates

Theorem 2.27 - Local compactness
Let $\Sigma_i^k \to \mathbb{R}^n$ *be a sequence of minimal immersions with a uniform bound*

$$|A^{\Sigma_i}| \le C.$$

Suppose that $p_i \in \Sigma_i$ *are points such that the images of* p_i *converge to* $p \in \mathbb{R}^n$ *and* $d_{\Sigma_i}(p_i, \partial\Sigma_i) \ge r_0 > 0$. *Then, after passing to a subsequence, local sheets of* Σ_i *through* p_i *converge smoothly to a minimal submanifold through* p. *In the embedded case one can also formulate the convergence in terms of minimal laminations.*

Proof. After a rotation, take the limiting tangent plane to be $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$. A uniform bound on $|A^{\Sigma_i}|$, together with the distance bound from the boundary, implies that the local sheets through p_i are graphs

$$F_i \colon B^k_\rho(0) \longrightarrow \mathbb{R}^{n-k}$$

for a fixed small $\rho > 0$, with uniformly bounded C^2 norms. By Arzelà–Ascoli, after passing to a subsequence,

$$F_i \longrightarrow F \qquad \text{in } C^{1,\alpha}$$

on a smaller ball. Since Σ_i is minimal, the functions F_i solve a uniformly elliptic system. Interior elliptic estimates and bootstrapping give

$$F_i \longrightarrow F \qquad \text{smoothly on compact subsets.}$$

□

Theorem 2.28 - Curvature estimate
There exists $C = C(n) > 0$ such that, if $\Sigma^n \to \mathbb{R}^{n+1}$, $2 \le n \le 6$, is a two-sided stable minimal immersion, complete up to its nonempty boundary, then

$$|A^\Sigma|(p)\, d_\Sigma(p, \partial\Sigma) \le C \qquad \textit{for every } p \in \Sigma.$$

If $\partial\Sigma = \varnothing$ and Σ is complete, the stable Bernstein theorem gives $A \equiv 0$. For $n = 6$ we use the 2026 stable Bernstein result [29]. An analogous local estimate holds in Riemannian manifolds of bounded geometry, with a constant also depending on the ambient geometry and a fixed radius scale.

Proof. Suppose by contradiction that there are stable minimal hypersurfaces Σ_i and points $p_i \in \Sigma_i$ such that

$$|A^{\Sigma_i}|(p_i)\, d_{\Sigma_i}(p_i, \partial\Sigma_i) \longrightarrow \infty.$$

Using a point-picking argument, we may replace p_i by points q_i and set $\lambda_i = |A^{\Sigma_i}|(q_i)$ so that, after translation and rescaling by λ_i,

$$q_i = 0, \qquad |A^{\Sigma_i}|(0) = 1,$$

while on intrinsic balls $B^{\Sigma_i}_{R_i}(0)$,

$$\sup_{B^{\Sigma_i}_{R_i}(0)} |A^{\Sigma_i}| \le 2, \qquad R_i \longrightarrow \infty.$$

By local compactness and a diagonal argument, a subsequence converges smoothly on compact subsets to a complete minimal immersion

$$\Sigma^n \longrightarrow \mathbb{R}^{n+1}$$

with $|A^\Sigma|(0) = 1$. Stability is preserved under smooth convergence, so the limit is stable. This contradicts the stable Bernstein theorem. Hence the estimate follows. □

2.12 Structure of finite-index minimal surfaces

For now, let $\Sigma^2 \subset \mathbb{R}^3$ be a complete, two-sided minimal surface. Recall that

$$L_\Sigma = \Delta_\Sigma + |A|^2, \qquad Q^\Sigma(u,u) = \int_\Sigma (|\nabla u|^2 - |A|^2 u^2)\, \mathrm{d}\,\mathrm{vol}_\Sigma\,.$$

Definition 2.29
The *index* of Σ is

$$\operatorname{ind}(\Sigma) = \lim_{i\to\infty} \operatorname{ind}(\Omega_i),$$

where $\Omega_1 \Subset \Omega_2 \Subset \cdots \subset \Sigma$ is an exhaustion by relatively compact open sets. The limit does not depend on the exhaustion.

Example 2.13
The plane has index 0, the catenoid has index 1, and the helicoid has infinite index.

Lemma 2.30 - Dirichlet spectrum
Let $\Omega \subset \Sigma$ be a connected, relatively compact domain with smooth boundary. The Jacobi operator $L_\Sigma = \Delta_\Sigma + |A|^2$, with Dirichlet boundary conditions, has discrete spectrum

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_I < 0 \le \lambda_{I+1} \le \cdots \longrightarrow \infty$$

for $-L_\Sigma$, where $I = \operatorname{ind}(\Omega)$ and eigenvalues are counted with multiplicity. The associated eigenfunctions $\varphi_1, \varphi_2, \ldots$ form an orthonormal basis of $L^2(\Omega)$. Moreover, λ_1 is simple and φ_1 can be chosen strictly positive in Ω.

We have the variational characterization

$$\begin{aligned}\lambda_1(-L_\Sigma, \Omega) &= \inf_{\substack{u\in C_c^\infty(\Omega)\\ u\neq 0}} \frac{-\int_\Omega u L_\Sigma u \,\mathrm{d}\operatorname{vol}_\Sigma}{\int_\Omega u^2 \,\mathrm{d}\operatorname{vol}_\Sigma} \\ &= \inf_{\substack{u\in W_0^{1,2}(\Omega)\\ u\neq 0}} \frac{Q^\Sigma(u,u)}{\int_\Omega u^2 \,\mathrm{d}\operatorname{vol}_\Sigma}.\end{aligned}$$

If u attains this infimum, then u is smooth and satisfies

$$-L_\Sigma u = \lambda_1 u \quad \text{on } \Omega, \qquad u|_{\partial\Omega} = 0.$$

Proposition 2.31 - Fischer-Colbrie–Schoen [22]; compare Barta [2]
Let $\Omega \subset \Sigma$ be open and connected. Then Ω is stable if and only if there exists $u \in C^\infty(\Omega)$ such that

$$u > 0 \quad \text{and} \quad L_\Sigma u = 0 \quad \text{in } \Omega.$$

Proof. Suppose first that $u > 0$ and $L_\Sigma u = 0$, and set $w = \log u$. Then

$$\nabla w = \frac{\nabla u}{u}, \qquad \Delta_\Sigma w = \frac{\Delta_\Sigma u}{u} - |\nabla w|^2 = -|A|^2 - |\nabla w|^2.$$

For $\varphi \in C_c^\infty(\Omega)$, integration by parts gives

$$\begin{aligned}\int_\Omega \varphi^2 |A|^2 \,\mathrm{d}\operatorname{vol}_\Sigma &= -\int_\Omega (\Delta_\Sigma w + |\nabla w|^2)\varphi^2 \,\mathrm{d}\operatorname{vol}_\Sigma \\ &= \int_\Omega \left(2\varphi\langle \nabla w, \nabla\varphi\rangle - |\nabla w|^2\varphi^2\right) \mathrm{d}\operatorname{vol}_\Sigma \\ &\le \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}\operatorname{vol}_\Sigma .\end{aligned}$$

Thus Ω is stable. This is the implication related to Barta's principle.

Conversely, suppose that Ω is stable. Choose an exhaustion by smooth connected domains

$$\Omega_1 \Subset \Omega_2 \Subset \cdots \Subset \Omega$$

and fix $x_0 \in \Omega_1$. By stability and strict domain monotonicity of the first Dirichlet eigenvalue,

$$0 \leq \lambda_1(-L_\Sigma, \Omega_{i+1}) < \lambda_1(-L_\Sigma, \Omega_i).$$

Standard elliptic theory gives v_i solving

$$\begin{cases} L_\Sigma v_i = 0 & \text{in } \Omega_i, \\ v_i = 1 & \text{on } \partial\Omega_i. \end{cases}$$

By the maximum principle, $v_i > 0$. Set

$$u_i(x) = \frac{v_i(x)}{v_i(x_0)}, \qquad u_i(x_0) = 1.$$

For every $K \Subset \Omega$, the Harnack inequality gives a constant $C = C(K)$ such that

$$\sup_K u_i \leq C \inf_K u_i \leq C$$

for i large enough (using a chain of compact sets joining K to x_0). Interior elliptic estimates, followed by the Schauder estimates, give uniform $C^{k,\alpha}$ bounds on $K' \Subset \Omega$. Hence, up to a subsequence,

$$u_i \longrightarrow u \quad \text{in } C^\infty_{\text{loc}}(\Omega),$$

where $L_\Sigma u = 0$ and $u(x_0) = 1$. The strong maximum principle yields $u > 0$ in Ω. □

Recall that the topology of a noncompact surface is described in particular by its genus and its ends. An end is represented by a nested sequence of unbounded connected components of the complements of compact subsets. For example, the helicoid has genus 0 and one end.

Theorem 2.32 - Structure of finite-index minimal surfaces [21]
Let $\Sigma^2 \subset \mathbb{R}^3$ be a complete, connected, two-sided, embedded minimal surface. The following are equivalent:

(i) $\operatorname{ind}(\Sigma) < \infty$*;*

(ii) Σ *is stable outside a compact set;*

(iii) Σ *has finite total curvature, i.e.*

$$\int_\Sigma |K| \, \mathrm{d}\,\mathrm{vol}_\Sigma = \frac{1}{2} \int_\Sigma |A|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma < \infty.$$

If these conditions hold, then:

- Σ *is conformally equivalent to*

$$\overline{\Sigma} \setminus \{p_1, \ldots, p_e\},$$

where $\overline{\Sigma}$ is a compact Riemann surface of genus g and $e \geq 1$ is the number of ends.

- *The Gauss map extends to a smooth, anticonformal map*

$$\overline{N}\colon \overline{\Sigma} \longrightarrow \mathbb{S}^2,$$

which is either constant (for a plane) or a branched covering of degree $d \geq 1$. Moreover,

$$\int_\Sigma (-K)\,\mathrm{d}\,\mathrm{vol}_\Sigma = 4\pi d.$$

If h is any smooth metric in the conformal class of $\overline{\Sigma}$, then $\mathrm{ind}(\Sigma)$ *equals the number of negative eigenvalues of*

$$-\Delta_h - |\mathrm{d}\overline{N}|_h^2 = -\Delta_h - 2|\det \mathrm{d}\overline{N}|_h$$

on $\overline{\Sigma}$, counted with multiplicity.

- *For $C \subset \Sigma$ sufficiently large and compact, $\Sigma \setminus C$ has exactly e connected components. Each end is asymptotic to a plane or to an end of a catenoid; after rotation, it is a graph over $\{|w| > R\} \subset \mathbb{R}^2$ of the form*

$$z = \alpha_j \log|w| + \beta_j + O(|w|^{-1}),$$

where $\alpha_j = 0$ for a planar end and $\alpha_j \neq 0$ for a catenoidal end. With a consistent orientation of the ends, $\sum_j \alpha_j = 0$. In addition, the Jorge–Meeks formula [31] gives

$$\int_\Sigma K\,\mathrm{d}\,\mathrm{vol}_\Sigma = -4\pi(g + e - 1).$$

Proof. We first prove the equivalence of the three conditions.

(i) ⇒ (ii). Let $I = \mathrm{ind}(\Sigma) < \infty$ and let $W \subset C_c^\infty(\Sigma)$ be an I-dimensional subspace where Q^Σ is negative definite. There is a compact set $C \subset \Sigma$ containing the supports of a basis of W. If $\Sigma \setminus C$ were not stable, there would be $\varphi \in C_c^\infty(\Sigma \setminus C)$ with $Q^\Sigma(\varphi, \varphi) < 0$. Since the supports are disjoint,

$$Q^\Sigma(w + s\varphi, w + s\varphi) = Q^\Sigma(w, w) + s^2 Q^\Sigma(\varphi, \varphi) < 0$$

for every nonzero $w + s\varphi \in W \oplus \langle\varphi\rangle$. This contradicts the definition of the index.

(ii) ⇒ (iii). Take a compact set C such that $\Sigma \setminus C$ is stable. By the preceding proposition there is $u > 0$, smooth on Σ, such that

$$L_\Sigma u = 0 \quad \text{on } \Sigma \setminus C.$$

Here u has been smoothly extended across C, keeping it strictly positive.

A conformal change of metric. Consider

$$\widetilde{g} = u^2 g_\Sigma.$$

Outside C, its Gaussian curvature satisfies

$$\begin{aligned}\widetilde{K} &= \frac{1}{u^2}(K - \Delta_\Sigma \log u)\\ &= \frac{1}{u^2}\left(K - \frac{u\Delta_\Sigma u - |\nabla u|^2}{u^2}\right)\\ &= \frac{1}{u^2}\left(K + |A|^2 + \frac{|\nabla u|^2}{u^2}\right)\\ &= \frac{|A|^2}{2u^2} + \frac{|\nabla u|^2}{u^4} \geq 0,\end{aligned}$$

where we used $K = -|A|^2/2$.

Completeness of $\widetilde{g}$. We recall the argument of Fischer-Colbrie [21]. Fix $x_0 \in \Sigma$. For large R take a smooth nonnegative function ε_R such that

$$\varepsilon_R = 0 \text{ on } B_R^\Sigma(x_0), \qquad \varepsilon_R = 1 \text{ outside } B_{R+1}^\Sigma(x_0),$$

and set $u_R = u + \varepsilon_R$. The metric $\widetilde{g}_R = u_R^2 g_\Sigma$ is complete, since g_Σ is complete and $u_R \geq 1$ outside a compact set. Take a $\widetilde{g}_R$-minimizing geodesic from x_0 to the complement of $B_R^\Sigma(x_0)$. A subsequence converges locally to an escaping $\widetilde{g}$-minimizing geodesic $\gamma\colon [0,\infty) \to \Sigma$, parametrized by g_Σ-arclength s.

It is enough to show that

$$\int_0^\infty u(\gamma(s))\,\mathrm{d}s = \infty.$$

Indeed, $\mathrm{d}\widetilde{s} = u\,\mathrm{d}s$. The second variation of length along a minimizing segment of γ (cf. (1.1)) gives

$$\int_0^\infty \left(\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\widetilde{s}}\right)^2 - \widetilde{K}\varphi^2\right)\mathrm{d}\widetilde{s} \geq 0$$

for compactly supported φ vanishing at the initial point, where the part near C is handled by taking the initial point outside C. Since $|\nabla u|^2 \geq (u')^2$, we obtain

$$\begin{aligned}\int_0^\infty \frac{(\varphi')^2}{u}\,\mathrm{d}s &\geq \int_0^\infty \left(\frac{|A|^2}{2u} + \frac{|\nabla u|^2}{u^3}\right)\varphi^2\,\mathrm{d}s\\ &\geq \int_0^\infty \frac{(u')^2}{u^3}\varphi^2\,\mathrm{d}s.\end{aligned}$$

Taking $\varphi = u\psi$ and expanding, the $(u')^2\psi^2/u$ terms cancel, and we find

$$\int_0^\infty \left(u(\psi')^2 + 2u'\psi\psi'\right)\mathrm{d}s \geq 0.$$

An integration by parts yields

$$\int_0^\infty u\big((\psi')^2 + 2\psi\psi''\big)\,\mathrm{d}s \leq 0.$$

Take $\psi(s) = s\xi_R(s)$, where $\xi_R = 1$ on $[0,R]$, $\xi_R = 0$ on $[2R,\infty)$, and

$$|\xi_R'| \leq \frac{C}{R}, \qquad |\xi_R''| \leq \frac{C}{R^2}.$$

Then $\psi' = 1$, $\psi'' = 0$ on $[0, R]$, while the other terms are bounded on $[R, 2R]$. Consequently,

$$\int_0^R u(\gamma(s))\,\mathrm{d}s \le C \int_R^{2R} u(\gamma(s))\,\mathrm{d}s.$$

If $\int_0^\infty u\,\mathrm{d}s$ were finite, the right-hand side would converge to zero as $R \to \infty$, whereas the left-hand side would not. Thus the metric $\widetilde{g}$ is complete.

Finite topology. We show next that Σ has finite genus and finitely many ends. Let

$$\ell(t) = |\partial B_t^{\widetilde{g}}(x_0)|_{\widetilde{g}}.$$

For a regular distance radius, the first variation of length and Gauss–Bonnet give, when the boundary is smooth,

$$\ell'(t) = \int_{\partial B_t^{\widetilde{g}}(x_0)} \kappa_g = 2\pi\chi(B_t^{\widetilde{g}}(x_0)) - \int_{B_t^{\widetilde{g}}(x_0)} \widetilde{K}\,\mathrm{d}\,\mathrm{vol}_{\widetilde{g}}.$$

For arbitrary radii the distance circles may have corners, and the Fiala–Hartman inequality gives instead, for almost every t,

$$\ell'(t) \le 2\pi\chi(B_t^{\widetilde{g}}(x_0)) - \int_{B_t^{\widetilde{g}}(x_0)} \widetilde{K}\,\mathrm{d}\,\mathrm{vol}_{\widetilde{g}}.$$

Since $\widetilde{K} \ge 0$ outside C, its negative part is supported in a compact set. Thus

$$\ell'(t) \le 2\pi\chi(B_t^{\widetilde{g}}(x_0)) + C_0.$$

If the topology were infinite, the Euler characteristic of large geodesic balls would become arbitrarily negative; the inequality would contradict $\ell(t) \ge 0$. Therefore Σ has finite topology.

The ends are punctures. By the classification of finite-topology surfaces, Σ is conformally equivalent to a compact Riemann surface with finitely many closed disks or points removed. We show that the ends correspond to points. In a topological annular end E, denote by $\ell_E(t)$ the total length, in $\widetilde{g}$, of the distance curve at distance t from a fixed inner boundary. The preceding length estimate gives

$$\ell_E(t) \le C(1 + t)$$

for large t. In particular the end admits *logarithmic cutoffs* with energy tending to zero. Indeed, if ρ is the $\widetilde{g}$-distance to the inner boundary, define

$$\eta_R(\rho) = \begin{cases} 1, & \rho \le R, \\ 1 - \dfrac{\log(\rho/R)}{\log R}, & R < \rho < R^2, \\ 0, & \rho \ge R^2. \end{cases}$$

The coarea formula gives

$$\int_E |\nabla^{\widetilde{g}} \eta_R|_{\widetilde{g}}^2\,\mathrm{d}\,\mathrm{vol}_{\widetilde{g}} \le \frac{C}{(\log R)^2} \int_R^{R^2} \frac{\ell_E(t)}{t^2}\,\mathrm{d}t \le \frac{C}{\log R} \longrightarrow 0.$$

The Dirichlet integral is conformally invariant in dimension two. A conformal annulus $\{r < |z| < 1\}$ with $r > 0$ has positive capacity between its boundary components, contrary to these cutoffs. Hence $r = 0$: every end is conformally a punctured disk.

Alternatively, writing $\widetilde{g} = \lambda(z)|\mathrm{d}z|^2$ on an end, the condition $\widetilde{K} \geq 0$ gives

$$\Delta_{\mathbb{R}^2} \log \frac{1}{\lambda} = 2\lambda \widetilde{K} \geq 0.$$

Set $w = \log(1/\lambda)$ and consider its circular averages

$$m(\rho) = \frac{1}{2\pi} \int_0^{2\pi} w(\rho e^{i\theta})\, \mathrm{d}\theta, \qquad s = \log \rho.$$

Then

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} m(e^s) = \frac{e^{2s}}{2\pi} \int_0^{2\pi} \Delta_{\mathbb{R}^2} w(e^s e^{i\theta})\, \mathrm{d}\theta \geq 0.$$

Thus m is convex in $\log \rho$. For $r < \rho < \rho_0 < \rho_1$,

$$m(\rho) \geq m(\rho_0) - \frac{m(\rho_1) - m(\rho_0)}{\log \rho_1 - \log \rho_0} \log \frac{\rho_0}{\rho},$$

so, if $r > 0$, $m(\rho)$ stays bounded below as $\rho \downarrow r$. In the alternative analytic proof, a conformal-factor estimate derived from completeness (of Cheng–Yau type) forces $m(\rho) \to -\infty$ as $\rho \downarrow r$, a contradiction. The capacity proof above establishes the puncture claim directly, without invoking this additional estimate.

We have now proved that

$$\Sigma \simeq \overline{\Sigma} \setminus \{p_1, \ldots, p_e\}.$$

Finite total curvature from stability. We retain the cutoff argument at the punctures. Choose a compact set $C' \supset C$ such that $\Sigma \setminus C'$ is a disjoint union of punctured disks $D_j \setminus \{p_j\}$. Let ϕ be a smooth function equal to 0 near C and to 1 on the ends outside C'. In the punctured-disk coordinate z about $p_j = 0$, let

$$\eta_\varepsilon(z) = \begin{cases} 0, & |z| \leq \varepsilon, \\ \dfrac{|z| - \varepsilon}{\varepsilon}, & \varepsilon < |z| < 2\varepsilon, \\ 1, & |z| \geq 2\varepsilon. \end{cases}$$

Take the product of these cutoff functions over the finitely many ends. Stability gives

$$\begin{aligned} \int_{\Sigma \setminus C} |A|^2 \eta_\varepsilon^2 \phi^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma &\leq \int_{\Sigma \setminus C} |\nabla(\eta_\varepsilon \phi)|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma \\ &\leq 2 \int_{\Sigma \setminus C} \phi^2 |\nabla \eta_\varepsilon|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma + 2 \int_{\Sigma \setminus C} \eta_\varepsilon^2 |\nabla \phi|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma . \end{aligned}$$

By conformal invariance, in the z coordinate

$$\int_{\varepsilon < |z| < 2\varepsilon} |\nabla \eta_\varepsilon|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma = \int_{\varepsilon < |z| < 2\varepsilon} \frac{1}{\varepsilon^2} \, \mathrm{d}x \, \mathrm{d}y = 3\pi.$$

The remaining terms are supported on a fixed compact set. We obtain a bound independent of ε, and letting $\varepsilon \to 0$ gives

$$\int_{\Sigma \setminus C'} |A|^2 \, \mathrm{d}\,\mathrm{vol}_\Sigma < \infty.$$

Adding the finite integral on C' proves finite total curvature.

There is also a shorter way to finish once the completeness of $\widetilde{g}$ is known: by the Cohn–Vossen inequality [11], the positive part of $\widetilde{K}$ is integrable because its negative part is supported on a compact set. Since

$$\widetilde{K}\,\mathrm{d}\,\mathrm{vol}_{\widetilde{g}} = \left(\frac{1}{2}|A|^2 + |\nabla \log u|^2\right)\mathrm{d}\,\mathrm{vol}_\Sigma \quad \text{outside } C,$$

we again obtain $\int_\Sigma |A|^2 < \infty$.

(iii) ⇒ (i). By Huber–Osserman [30, 42], finite total curvature implies that Σ is conformally a compact Riemann surface minus finitely many points and that the Gauss map extends smoothly across these points. Choose a smooth metric h on the compactification $\overline{\Sigma}$. Then

$$Q^\Sigma(\varphi,\varphi) = \int_{\overline{\Sigma}} (|\nabla^h \varphi|^2 - |\mathrm{d}\overline{N}|_h^2 \varphi^2)\,\mathrm{d}\,\mathrm{vol}_h$$

for all $\varphi \in C_c^\infty(\Sigma)$. The operator

$$-\Delta_h - |\mathrm{d}\overline{N}|_h^2$$

has discrete spectrum and only finitely many negative eigenvalues on $\overline{\Sigma}$. This gives $\mathrm{ind}(\Sigma) < \infty$. In fact the indices agree, since points have zero $W^{1,2}$-capacity on a surface: the usual logarithmic cutoff functions around each puncture have Dirichlet energy tending to zero.

Finally, recall from the Weierstrass representation that the Gauss map is anticonformal away from branch points. The finite-spherical-area condition excludes essential singularities at the punctures (one may also use the big Picard theorem); hence N extends to $\overline{N}\colon \overline{\Sigma} \to \mathbb{S}^2$. When N is nonconstant, $\overline{N}$ is a branched covering and

$$\int_\Sigma |K|\,\mathrm{d}\,\mathrm{vol}_\Sigma = \int_\Sigma |\det \mathrm{d}N|\,\mathrm{d}\,\mathrm{vol}_\Sigma = \int_{\mathbb{S}^2} \#\big(\overline{N}^{-1}(y)\big)\,\mathrm{d}\,\mathrm{vol}_{\mathbb{S}^2}(y) = 4\pi \deg \overline{N}.$$

The description of the ends and the Jorge–Meeks formula follow from the structure theory for embedded finite-total-curvature minimal surfaces [31]. □

2.13 Exercises

Exercise 2.14 - Catenoids
For all $\lambda > 0$, define $C_\lambda \subset \mathbb{R}^3$ to be the following surface of revolution, called *catenoid*,

$$C_\lambda \coloneqq \left\{(u_1,u_2,z) \in \mathbb{R}^3 \;:\; \lambda \cosh(z/\lambda) = \sqrt{u_1^2+u_2^2}\right\}.$$

(i) Prove that C_λ is a minimal surface for all $\lambda > 0$.

(ii) Show that there exists a unique value $\lambda > 0$ such that C_λ meets the Euclidean sphere $\partial B(0,1)$ orthogonally. The surface C_λ for this choice of λ is called *critical catenoid*.

(iii) Compute the Gaussian curvature K of the catenoid C_λ, and the total curvature $\int_{C_\lambda} |K| d\sigma$ for all $\lambda > 0$..

Exercise 2.15 - Radially symmetric minimal surfaces
Let $\Omega \subset \mathbb{R}^2$ be a connected open set and take a C^2 function $f\colon \Omega \to \mathbb{R}$. Assume that both Ω and f are radially symmetric, and write $f(x) = w(|x|)$ for some C^2 function $w\colon I \to \mathbb{R}$, where $I \subset \mathbb{R}_+$ is an interval.

(i) Prove that graph(f) is minimal if and only if

$$rw''(r) + w'(r)(1 + (w'(r))^2) = 0 \quad \text{for all } r \in I. \qquad \text{(MSE : } ar\text{)}$$

(ii) Show that, if (MSE : ar) holds for $r \in [a, b]$, then

$$\log\left(\frac{w'(r)}{\sqrt{1 + (w'(r))^2}}\right) - \log\left(\frac{w'(a)}{\sqrt{1 + (w'(a))^2}}\right) = \log\left(\frac{a}{r}\right)$$

for all $r \in [a, b]$.

(iii) Determine all radially symmetric solutions to the minimal surface equation defined on the following domains $U \subset \mathbb{R}^2$:

- an annulus of radii a, b, namely $\Omega = \{x \in \mathbb{R}^2 \ : \ a < |x| < b\}$;
- the complement of a ball of radius a, namely $\Omega = \{x \in \mathbb{R}^2 \ : \ |x| > a\}$;
- the whole Euclidean plane $\Omega = \mathbb{R}^2$.

Exercise 2.16 - Ellipticity of minimal surface equation
Recall that the minimal surface equation for the graph of a function $u\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ can be written as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \Delta u - \frac{D^2u(\nabla u, \nabla u)}{1 + |\nabla u|^2} = \sum_{i,j=1}^{n} a_{ij} D^2_{ij} u = 0,$$

where

$$a_{ij} \coloneqq \delta_{ij} - \frac{D_i u D_j u}{1 + |\nabla u|^2}.$$

Prove that $(a_{ij})_{i,j=1,\dots,n}$ is a positive definite matrix with eigenvalues 1 with multiplicity $n - 1$ and $(1 + |\nabla u|^2)^{-1}$ with multiplicity 1. Conclude that the minimal surface equation is not uniformly elliptic.

Exercise 2.17 - The maximum principle
The goal of this exercise is to prove the *strong maximum principle for minimal hypersurfaces*, Theorem 2.16, in the case of surface in $\mathbb{R}^3$ (i.e., $n = 2$ in the statement).

(i) Show that the theorem follows from this local graphical statement. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected neighborhood of the origin. Suppose that $u_1, u_2\colon \Omega \to \mathbb{R}$ are smooth solutions of the minimal surface equation such that $u_1(0) = u_2(0)$ and $u_1 \leq u_2$. Then $u_1 = u_2$.

(ii) Let us denote by $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ the map

$$F(X) \coloneqq \frac{X}{(1 + |X|^2)^{1/2}}.$$

Prove that, for all $X \in \mathbb{R}^2$, the eigenvalues of the matrix $dF(X)$ are bounded between $(1 + |X|^2)^{-3/2}$ and $(1 + |X|^2)^{-1/2}$.

(iii) Prove that $\operatorname{div}(A(x)\nabla(u_2 - u_1)) = 0$ where the matrix $A(x)$ is given by

$$A(x) \coloneqq \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))dt.$$

(iv) Show that the eigenvalues of $A(x)$ are uniformly bounded (from above and from below) by positive constants only depending on $\max_{i=1,2} \sup_{x\in\Omega} |\nabla u_i(x)|$. Namely, there exists a constant $\mu > 0$ that depends on $\max_{i=1,2} \sup_{x\in\Omega} |\nabla u_i(x)|$ such that $\mu < \lambda_1(x) \leq \lambda_2(x) < 1/\mu$ for all $x \in \Omega$, where $\lambda_1(x), \lambda_2(x)$ are the eigenvalues of $A(x)$.

(v) Conclude that $u_1 = u_2$.

Exercise 2.18 - Graphicality
Let $\Omega \subset \mathbb{R}^2$ be a strictly convex, open, bounded, regular domain and let $\gamma \subset \mathbb{R}^3$ be a simple closed curve which is a C^2 graph over $\partial\Omega$. Moreover let $\Sigma \subset \mathbb{R}^3$ be a compact connected minimal surface with $\partial\Sigma = \gamma$. Prove that Σ is graphical over Ω.

Exercise 2.19 - Constant mean curvature surfaces
Let $\Sigma^2 \subset \mathbb{R}^3$ be a closed embedded surface in $\mathbb{R}^3$.

1. Show that $\Sigma = \partial\Omega$ for some open set $\Omega \subset \mathbb{R}^3$.

2. Prove that Σ is a critical point of the area, with respect to the variations that preserve the volume of Ω if and only if Σ has constant mean curvature (often abbreviated CMC).

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