Minimal surfaces – Fall 2026

Exercise sheet 1

Exercise 1 (Catenoids)**.** For all $\lambda > 0$, define $C_\lambda \subset \mathbb{R}^3$ to be the following surface of revolution, called *catenoid*,

$$C_\lambda := \left\{ (u_1, u_2, z) \in \mathbb{R}^3 \mid \lambda \cosh(z/\lambda) = \sqrt{u_1^2 + u_2^2} \right\}.$$

(i) Prove that C_λ is a minimal surface for all $\lambda > 0$.

(ii) Show that there exists a unique value $\lambda > 0$ such that C_λ meets the Euclidean sphere $\partial B(0,1)$ orthogonally. The surface C_λ for this choice of λ is called *critical catenoid*.

(iii) Compute the Gaussian curvature K of the catenoid C_λ, and the total curvature $\int_{C_\lambda} |K| d\sigma$ for all $\lambda > 0$..

Exercise 2 (Radially symmetric minimal surfaces)**.** Let $\Omega \subset \mathbb{R}^2$ be a connected open set and take a C^2 function $f \colon \Omega \to \mathbb{R}$. Assume that both Ω and f are radially symmetric, and write $f(x) = w(|x|)$ for some C^2 function $w \colon I \to \mathbb{R}$, where $I \subset \mathbb{R}_+$ is an interval.

(i) Prove that graph(f) is minimal if and only if

$$r w''(r) + w'(r)(1 + (w'(r))^2) = 0 \quad \text{for all } r \in I. \tag{MSE$\star$}$$

(ii) Show that, if (MSE$\star$) holds for $r \in [a, b]$, then

$$\log\left(\frac{w'(r)}{\sqrt{1 + (w'(r))^2}} \right) - \log\left(\frac{w'(a)}{\sqrt{1 + (w'(a))^2}} \right) = \log\left(\frac{a}{r} \right)$$

for all $r \in [a, b]$.

(iii) Determine all radially symmetric solutions to the minimal surface equation defined on the following domains $U \subset \mathbb{R}^2$:

- an annulus of radii a, b, namely $\Omega = \{x \in \mathbb{R}^2 \mid a < |x| < b\}$;
- the complement of a ball of radius a, namely $\Omega = \{x \in \mathbb{R}^2 \mid |x| > a\}$;
- the whole Euclidean plane $\Omega = \mathbb{R}^2$.

Exercise 3 (Ellipticity of minimal surface equation)**.** Recall that the minimal surface equation for the graph of a function $u \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$ can be written as

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \Delta u - \frac{D^2 u(\nabla u, \nabla u)}{1 + |\nabla u|^2} = \sum_{i,j=1}^{n} a_{ij} D^2_{ij} u = 0,$$

where

$$a_{ij} \coloneqq \delta_{ij} - \frac{D_i u D_j u}{1 + |\nabla u|^2}.$$

Prove that $(a_{ij})_{i,j=1,\dots,n}$ is a positive definite matrix with eigenvalues 1 with multiplicity $n-1$ and $(1+|\nabla u|^2)^{-1}$ with multiplicity 1. Conclude that the minimal surface equation is not uniformly elliptic.

Exercise 4 (The maximum principle)**.** The goal of this exercise is to prove the *strong maximum principle for minimal hypersurfaces* in the case of surface in $\mathbb{R}^3$ (i.e., $n = 2$ in the statement).

Theorem 5. *Let $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n+1}$ be two complete properly embedded minimal hypersurfaces in $\mathbb{R}^{n+1}$. Assume that $\Sigma_1 \cap \Sigma_2 \neq \emptyset$ and Σ_2 lies on one side of Σ_1. Then $\Sigma_1 = \Sigma_2$.*

(i) Show that the theorem follows from this local graphical statement. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected neighborhood of the origin. Suppose that $u_1, u_2 \colon \Omega \to \mathbb{R}$ are smooth solutions of the minimal surface equation such that $u_1(0) = u_2(0)$ and $u_1 \leq u_2$. Then $u_1 = u_2$.

(ii) Let us denote by $F \colon \mathbb{R}^2 \to \mathbb{R}^2$ the map

$$F(X) := \frac{X}{(1+|X|^2)^{1/2}}.$$

Prove that, for all $X \in \mathbb{R}^2$, the eigenvalues of the matrix $dF(X)$ are bounded between $(1+|X|^2)^{-3/2}$ and $(1+|X|^2)^{-1/2}$.

(iii) Prove that $\operatorname{div}(A(x)\nabla(u_2 - u_1)) = 0$ where the matrix $A(x)$ is given by

$$A(x) := \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))dt.$$

(iv) Show that the eigenvalues of $A(x)$ are uniformly bounded (from above and from below) by positive constants only depending on $\max_{i=1,2} \sup_{x\in\Omega} |\nabla u_i(x)|$. Namely, there exists a constant $\mu > 0$ that depends on $\max_{i=1,2} \sup_{x\in\Omega} |\nabla u_i(x)|$ such that $\mu < \lambda_1(x) \leq \lambda_2(x) < 1/\mu$ for all $x \in \Omega$, where $\lambda_1(x), \lambda_2(x)$ are the eigenvalues of $A(x)$.

(v) Conclude that $u_1 = u_2$.

Exercise 6 (Graphicality)**.** Let $\Omega \subset \mathbb{R}^2$ be a strictly convex, open, bounded, regular domain and let $\gamma \subset \mathbb{R}^3$ be a simple closed curve which is a C^2 graph over $\partial\Omega$. Moreover let $\Sigma \subset \mathbb{R}^3$ be a compact connected minimal surface with $\partial\Sigma = \gamma$. Prove that Σ is graphical over Ω.

Exercise 7 (Constant mean curvature surfaces)**.** Let $\Sigma^2 \subset \mathbb{R}^3$ be a closed embedded surface in $\mathbb{R}^3$.

1. Show that $\Sigma = \partial\Omega$ for some open set $\Omega \subset \mathbb{R}^3$.
2. Prove that Σ is a critical point of the area, with respect to the variations that preserve the volume of Ω if and only if Σ has constant mean curvature (often abbreviated CMC).