Minimal surfaces – Fall 2026

Exercise sheet 0

Exercise 1 (Mean curvature of graphs)**.** Let $f\colon \Omega \to \mathbb{R}$ be a smooth function with $\Omega \subset \mathbb{R}^n$ smooth open set. Let $\Sigma = \operatorname{graph} f \subset \mathbb{R}^{n+1}$ be the graph of f with metric tensor g inherited by $\mathbb{R}^{n+1}$.

(i) Show that Σ is a hypersurface in $\mathbb{R}^{n+1}$, by finding a global chart (Ω, φ) for Σ.

(ii) Denoting by ∂_i and ∇ the usual differentiation on $\mathbb{R}^n$, by N the upward unit normal field of Σ, and by II_N and H the second fundamental form and the mean curvature of Σ in direction N respectively. Prove that

$$g_{ij} = \delta_{ij} + \partial_i f \partial_j f, \qquad N = \frac{(-\nabla f, 1)}{\sqrt{1+|\nabla f|^2}}, \qquad \mathrm{II}_N = \frac{\nabla^2 f}{\sqrt{1+|\nabla f|^2}},$$

$$H = \operatorname{div}\left(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}\right) = \frac{\Delta f}{\sqrt{1+|\nabla f|^2}} - \frac{(\nabla^2 f)(\nabla f, \nabla f)}{\left(\sqrt{1+|\nabla f|^2}\right)^3}.$$

Exercise 2 (Two-sidedness of hypersurfaces)**.** Show that every properly embedded hypersurface Σ^n in $\mathbb{R}^{n+1}$ is two-sided.

Exercise 3 (Divergence and Laplacian)**.** Let (M^n, g) be a Riemannian manifold.

(i) Prove the expression of the divergence with respect to the metric in coordinates

$$\operatorname{div}(X) = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} X^k\right).$$

(ii) Prove the expression of the Laplacian with respect to the metric in coordinates

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \partial_k \left(\sqrt{\det(g_{ij})} g^{kl} \partial_l f\right).$$

(iii) Show that, if $f \in C_c^\infty(M)$ and $\Delta f \geq 0$, then f is constant.

Exercise 4 (Normal frame, normal chart, exponential normal chart)**.** Let us fix a Riemannian manifold (M^n, g) and a point $p \in M$. Let us set $\{(e_i)|_p\}_{i=1}^n$ an orthonormal basis of T_pM.

(i) Show that there exists a local orthonormal frame $\{e_i\}_{i=1}^n$ of TM around p such that $e_i(p) = (e_i)|_p$, and $(\nabla_{e_i} e_j)(p) = 0$ for every $1 \leq i, j \leq n$. Notice that then, in such a frame, the Christoffel symbols are zero at p, i.e., $\Gamma_{ij}^k(p) = 0$ for every $1 \leq i, j, k \leq n$. We will call such a frame a *normal frame* around p.

(ii) Show that there exists a chart (U, φ) around p such that $\partial_i(p) = (e_i)|_p$ and $(\nabla \partial_i)(p) = 0$ for every $1 \leq i \leq n$. Show that if in this chart we have $g = g_{ij} dx^i \otimes dx^j$, then $g_{ij}(p) = \delta_{ij}(p)$, $\partial_k g_{ij}(p) = 0$, and $\Gamma_{ij}^k(p) = 0$ for every $1 \leq i, j, k \leq n$. We will call such a chart a *normal chart* around p. Notice that one example of such a chart is the *exponential normal chart*, that gives rise to *exponential normal coordinates*.

Exercise 5 (Warped products $I \times_f N$)**.** Let (N^n, g) be a Riemannian manifold and let $I \subset \mathbb{R}$ be an interval. Given $f \in C^\infty(I)$ with $f \neq 0$ on I, we define the *warped product* manifold $M = I \times_f N$ to be the $(n+1)$-dimensional manifold $I \times N$ endowed with the warped metric

$$\overline{g} = dt^2 + f^2(t) g.$$

Let $e_1, \ldots, e_n$ be a local orthonormal frame on an open set $U \subset N$. Assume also that $\{e_i\}$ is normal at some $x \in U$. Let also

$$\overline{e}_0 = \partial_t, \qquad \overline{e}_i = \frac{1}{f} e_i.$$

(i) Observe that $\overline{e}_0, \overline{e}_1, \dots, \overline{e}_n$ is an orthonormal frame on $I \times U$.

(ii) If R and $\overline{R}$ are the Riemann tensor on N and $I \times_f N$ respectively, prove that

$$\overline{R}_{0ijk} = \overline{R}_{i0jk} = 0, \qquad \overline{R}_{i0j0} = -\frac{f''}{f}\delta_{ij}, \qquad \overline{R}_{ijkl} = \frac{1}{f^2}R_{ijkl} - \left(\frac{f'}{f}\right)^2(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

for any $i, j, k, l = 1, \dots, n$.

(iii) If N has constant sectional curvature $c \in \mathbb{R}$, deduce that

$$\overline{\mathrm{Sec}}(\nabla t \wedge \overline{e}_i) = -\frac{f''}{f}, \qquad \overline{\mathrm{Sec}}(\overline{e}_i \wedge \overline{e}_j) = \frac{c - (f')^2}{f^2}$$

for any $i, j = 1, \dots, n$ with $i \neq j$.

Exercise 6 (Radial models of constant sectional curvature)**.** A *radial model of constant sectional curvature* $c \in \mathbb{R}$ is

$$\big((0,T) \times \mathbb{S}^{n-1}, \langle\cdot,\cdot\rangle := \mathrm{d}r^2 + \mathrm{sn}_c(r)^2\,\mathrm{d}\theta^2\big),$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional sphere of radius one and $\mathrm{d}\theta^2$ is the standard metric on it; the function $\mathrm{sn}_c(r)$ is the positive solution of the Jacobi equation

$$\begin{cases} \mathrm{sn}_c''(r) + c\,\mathrm{sn}_c(r) = 0, \\ \mathrm{sn}_c(0) = 0, \\ \mathrm{sn}_c'(0) = 1; \end{cases}$$

and $T := \sup\{\varrho \mid \mathrm{sn}_c(r) > 0\ \forall r \in (0,\varrho)\}$.

(i) Prove that

$$\mathrm{sn}_c(r) = \begin{cases} \frac{1}{\sqrt{c}}\sin(\sqrt{c}\,r) & \text{if } c > 0, \text{ then } T = \frac{\pi}{\sqrt{c}}, \\ r & \text{if } c = 0, \text{ then } T = +\infty, \\ \frac{1}{\sqrt{-c}}\sinh(\sqrt{-c}\,r) & \text{if } c < 0, \text{ then } T = +\infty. \end{cases}$$

(ii) Show that, given $r(\cdot) := d(0,\cdot)$ the distance from the origin in this model, we have

$$\nabla^2 r(\cdot,\cdot) = \frac{\mathrm{sn}_c'(r)}{\mathrm{sn}_c(r)}\left(\langle\cdot,\cdot\rangle - \mathrm{d}r \otimes \mathrm{d}r(\,\cdot,\cdot)\right).$$

(iii) Show that these models have actually constant sectional curvature c.

Exercise 7 (The round sphere)**.** For $n \geq 2$, let $\mathbb{S}^n_r := \{x \in \mathbb{R}^{n+1} \mid |x| = r\}$ be the round sphere of radius r, with the metric g inherited by $\mathbb{R}^{n+1}$ and the outward unit normal $N(x) = x/r$.

(i) Show that $\mathbb{S}^n_r$ has second fundamental form $A(X,Y) = -\frac{1}{r}\langle X, Y\rangle N$, second fundamental form in the normal direction $\mathrm{II}(X,Y) = \frac{1}{r}\langle X, Y\rangle$, and shape operator $S_N = -\frac{1}{r}\mathrm{id}$. Deduce that the vector and scalar mean curvatures of $\mathbb{S}^n_r$ are equal to $\vec{H} = -\frac{n}{r}N$ and $H = \frac{n}{r}$.

(ii) Show that the curvature tensor of $\mathbb{S}^n_r$ is

$$R(X,Y,Z,W) = \frac{1}{r^2}\big(\langle X, Z\rangle\langle Y, W\rangle - \langle X, W\rangle\langle Y, Z\rangle\big).$$

Deduce that $\mathbb{S}^n_r$ has constant sectional curvature $\frac{1}{r^2}$, Ricci curvature $\mathrm{Ric} = \frac{n-1}{r^2}\,g$, and scalar curvature $\mathrm{Scal} = \frac{n(n-1)}{r^2}$.